THE SECOND DUAL OF THE SPACE OF CONTINUOUS FUNCTIONS. IV

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1. **Introduction.** In the present paper, we return to a compact space X. C denotes the Banach space of continuous (real) functions on X, and L and M, its first and second duals. C is actually a Banach lattice, and as with all Banach lattices, L is precisely the space $\Omega(C)$ of (order) bounded linear functionals on C; and similarly, M is precisely $\Omega(L)$.

Our continuing aim is the investigation of the structure of M and the manner in which C is imbedded in M. In pursuance of this, we single out, in this paper, two objects of study:

(1) Given a vector lattice E, the space $\tilde{\Omega}(E)$ of (order) continuous linear functionals is a closed ideal in $\Omega(E)$. It may, at one extreme, coincide with $\Omega(E)$, and at the other, consist only of 0. As we know [7[, $\tilde{\Omega}(M) = L$, which has received considerable study. About $\tilde{\Omega}(C)$, however, and its relationship to M, not too much has been known, and we make it one of our objects of study here.

(2) For each $f \in M$, let $u(f)$ denote the smallest u.s.c. element $\geqq f$, and $l(f)$ denote the largest l.s.c. element $\leqq f$. $u(f)$ and $l(f)$ correspond respectively to the closure and interior of a set in topology. We call an element f of M *rare* (corresponding to nowhere dense) if $l(u(|f|)) = 0$. The rare elements constitute a norm-closed ideal Ra in M. It is Ra which is our second principal object of study.

In §2, we establish the elementary properties of $u(f)$ and $l(f)$. In §3, we do the same for Ra. In §4, we start with the decomposition $L = \tilde{\Omega}(C) \oplus \tilde{\Omega}(Ra)$ and proceed to study the structure of M in terms of it. §§5, 7, 8, and 9 continue this study; in §9, we also discuss the cut-completion of C. In §10, we study the σ-closure of Ra; we denote it by Me and call its elements *meager* (the terms "rare" and "meager" are taken from Bourbaki). In §11, we consider the case where C itself is a complete vector lattice.

§§6 and 12 are devoted to the study and application of a (vector lattice) homomorphism of the type $h: C(X) \to C(Y)$, the resulting transpose mapping $h^t: L(Y) \to L(X)$, and the second transpose $h^{tt}: M(X) \to M(Y)$.

We want to comment here on the parallel between (M)-spaces and Boolean algebras. The Stone space of a Boolean algebra is totally disconnected, and in fact the Boolean algebra can be identified with the family of clopen (closed-and-open) subsets of the space. In contradistinction to this, the Kakutani-Stone

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space of an (M)-space is in general not totally disconnected. On the one hand, of course, we have a gain; every compact space X is the Kakutani-Stone space of some (M)-space, viz. C(X). On the other hand, it seems to indicate that a large part of the theory of Boolean algebras has no parallel in that of (M)-spaces.

But now consider the following. In the parallel between functions and sets, upper semicontinuous functions correspond to closed sets and lower semicontinuous functions to open sets. It follows continuous functions correspond to clopen sets. Thus, for example, the basic property of a totally disconnected compact space, that every open neighborhood of a closed set contains a clopen neighborhood of the set, finds its parallel in the following property of any compact space (actually of more general spaces): between every upper semicontinuous function and lower semicontinuous function lying above it, there is always a continuous function. So after all, in some sense, "total disconnectedness" is still with us. Perhaps we can have our cake and eat it too.

As a result of these heuristic considerations, we suggest the following as a fertile intuitive tool for carrying over properties of Boolean algebras to (M)-spaces: to consider M as the set of "bounded functions" on some totally disconnected compact base space, and to consider the imbedding of C in M as the "representation" of C on the set of "clopen functions" of the base space—a base space which actually does not exist. Theorem (11.1) below, for example, is better understood in this light.

At the request of the journal, we have omitted, throughout the paper, proofs which involve only routine calculations.

2. **The operations u, l, and δ.** The unmodified term "convergence" always means order-convergence, and unless otherwise indicated, the terms "limit," "closure," "continuous linear functional," etc., are with respect to this (order) convergence.

I and J always denote ideals. If I is a closed ideal in a complete vector lattice E, then we denote the projection in I of a subset A of E by A_I; in particular, the component in I of $a \in E$ is denoted by a_I. If I is the closed ideal generated by a single element b, then we may write A_b and a_b in place of A_I and a_I. Note that then $E_b = I$.

For typographical reasons, $\bar{A}_I$ will denote the (order) closure of A_I, not $(\bar{A})_I$. Similarly, if a set is denoted by two letters, such as Ra, we will denote its closure by a bar over the first letter only: $\bar{R}a$.

We define operations $u: M \to S$, $l: M \to S$ by

$$(2.1) \qquad u(f) = \bigwedge\{g \mid g \in C,\ g \geqq f\},\quad l(f) = \bigvee\{g \mid g \in C, g \leqq f\}.$$

In short, $u(f)$ is the smallest u.s.c. element $\geqq f$ and $l(f)$ is the largest l.s.c. element $\leqq f$. They correspond to the closure and interior of a set in topology. We develop the arithmetic of these two operations.

Clearly, f is u.s.c. if and only if $f = u(f)$, f is l.s.c. if and only if $f = l(f)$, and $f \in C$ if and only if $f = u(f) = l(f)$.

The following properties are immediate:

(2.2) (a) $f \leqq g$ *implies* $l(f) \leqq l(g)$, $u(f) \leqq u(g)$.
(b) *If* λ *is a non-negative real number, then* $u(\lambda f) = \lambda u(f)$ *and* $l(\lambda f) = \lambda l(f)$.
(c) $u(-f) = -l(f)$.

Almost as immediate are

(2.3) *For any bounded set* A *of* M,
(a) $u(\bigwedge_{f \in A} f) \leqq \bigwedge_{f \in A} u(f) \leqq \bigvee_{f \in A} u(f) \leqq u(\bigvee_{f \in A} f)$.
(b) $l(\bigwedge_{f \in A} f) \leqq \bigwedge_{f \in A} l(f) \leqq \bigvee_{f \in A} l(f) \leqq l(\bigvee_{f \in A} f)$.

For a finite number of elements, the last inequality in (a) and the first in (b) become equalities:

$$\text{(2.4)} \qquad \begin{aligned} u(f) \vee u(g) &= u(f \vee g), \\ l(f) \wedge l(g) &= l(f \wedge g), \end{aligned}$$

$$\text{(2.5)} \qquad l(f) + l(g) \leqq l(f+g) \leqq l(f) + u(g) \leqq u(f+g) \leqq u(f) + u(g).$$

As a corollary, we obtain

$$\text{(2.6)} \qquad l(f) - u(g) \leqq l(f-g) \leqq \begin{matrix} u(f) - u(g) \\ l(f) - l(g) \end{matrix} \leqq u(f-g) \leqq u(f) - l(g).$$

The following corresponds to a standard property of an open set.

(2.7) *If* g *is l.s.c., then for any* $f \in M$, $u(f) \wedge g \leqq u(f \wedge g)$.

Proof. $u(f) - u(f) \wedge g = (u(f) - g)^+ \geqq (f - g)^+ = f - f \wedge g \geqq f - u(f \wedge g)$, whence
(i) $u(f \wedge g) + u(f) - u(f) \wedge g \geqq f$.
Now from the first equality, $u(f) - u(f) \wedge g$ is u.s.c., hence, also the left side of (i). It follows we can replace the right side of (i) by $u(f)$, and we have the desired result.

For an arbitrary $g \in M$, if we let $l(g)$ be the g in (2.7), we have $u(f) \wedge l(g) \leqq u(f \wedge l(g)) \leqq u(f \wedge g)$. This gives the

Corollary. *Given* $f, g \in M$, *if* $f \wedge g = 0$, *then* $u(f) \wedge l(g) = 0$.

(2.7) also gives us the following, which we will need later.

(2.8) *Given a family* $\{g_\alpha\}$ *of l.s.c. elements, for any* $f \in M$, *we have*

$$u\left[\bigvee_\alpha l(u(f \wedge g_\alpha))\right] = u\left(l\left(u\left(\bigvee_\alpha f \wedge g_\alpha\right)\right)\right).$$

Proof. For any family $\{f_\alpha\}$, $\bigvee_\alpha l(u(f_\alpha)) \leqq l(\bigvee_\alpha u(f_\alpha)) \leqq l(u(\bigvee_\alpha f_\alpha))$ (2.3); hence (i) $u[\bigvee_\alpha l(u(f_\alpha))] \leqq u(l(u(\bigvee_\alpha f_\alpha)))$.

Thus, to establish (2.8), we need only show that the left side $\geqq$ the right side. From (2.7), $u(f \wedge g_\alpha) \geqq u(f) \wedge g_\alpha \geqq l(u(f)) \wedge g_\alpha$. Since this last is l.s.c., we obtain $l(u(f \wedge g_\alpha)) \geqq l(u(f)) \wedge g_\alpha$, hence $l(u(f \wedge g_\alpha)) \geqq l(u(f)) \wedge f \wedge g_\alpha$, hence $\bigvee_\alpha l(u(f \wedge g_\alpha)) \geqq l(u(f)) \wedge (\bigvee_\alpha f \wedge g_\alpha)$. Applying (2.7) again, $u[\bigvee_\alpha l(u(f \wedge g_\alpha))] \geqq l(u(f)) \wedge u(\bigvee_\alpha f \wedge g_\alpha) \geqq l(u(f)) \wedge l(u(\bigvee_\alpha f \wedge g_\alpha)) = l(u(\bigvee_\alpha f \wedge g))$. Since the first member of this last chain of inequalities is u.s.c., we can apply u to the last member, thus completing the proof.

Propositions (2.4) through (2.7) can be simplified if one of the elements involved is in C:

(2.9) *If* $g \in C$, *then for any* $f \in M$, $u(f \vee g) = u(f) \vee g$, $u(f \wedge g) = u(f) \wedge g$, $u(f + g) = u(f) + g$; *and similarly with u replaced by l.*

These are easily verified either by using the above propositions or the original definitions of u and l. Setting $g = 0$ in (2.9), we obtain

(2.10) (a) $(u(f))^+ = u(f^+)$,
(b) $(l(f))^+ = l(f^+)$,
(c) $(u(f))^- = l(f^-)$,
(d) $(l(f))^- = u(f^-)$.

Thus,

(2.11) (a) $u(f) = u(f^+) - l(f^-)$,
(b) $|u(f)| = u(f^+) + l(f^-)$,
(c) $l(f) = l(f^+) - u(f^-)$,
(d) $|l(f)| = l(f^+) + u(f^-)$.

This gives in turn,

$$l(|f|) \leqq \begin{matrix} |u(f)| \\ |l(f)| \end{matrix} \leqq u(|f|). \tag{2.12}$$

Proof. $|f| = f^+ + f^- \leqq u(f^+) + f^-$. Applying (2.5), $l(|f|) \leqq u(f^+) + l(f^-) = |u(f)|$. The other inequalities are shown similarly.

REMARK. It is not hard to show that actually $u(|f|) = |u(f)| \vee |l(f)|$.

(2.13) *Given* $f, g \in M$,

$$\begin{matrix} |u(f) - u(g)| \\ |l(f) - l(g)| \end{matrix} \leqq u(|f - g|).$$

Given $f \in M$, consider the two numbers $\lambda_1 = \inf\{\lambda \mid \lambda 1 \geqq f\}$, $\lambda_2 = \sup\{\lambda \mid \lambda 1 \leqq f\}$. Then $\|f\| = \max(|\lambda_1|, |\lambda_2|)$. Now $\lambda 1 \geqq f$ if and only if $\lambda 1 \geqq u(f)$, and $\lambda 1 \leqq f$ if and only if $\lambda 1 \leqq l(f)$. Combining this with (c) of (2.2) and the fact that $\|f\| = \|-f\|$, we obtain

(2.14) *For any* $f \in M$, $\|f\| = \max(\|u(f)\|, \|l(f)\|)$. *In particular, for* $f \geqq 0$, $\|f\| = \|u(f)\|$.

Combining this in turn with (2.13), we have the

COROLLARY. *Given* $f, g \in M$,

$$\begin{matrix} \|u(f) - u(g)\| \\ \|l(f) - l(g)\| \end{matrix} \leqq \|f - g\|.$$

Thus, the operations u and l are norm-continuous.

Given $f \in M$, we will denote $u(f) - l(f)$ by $\delta(f)$. $\delta(f)$ corresponds to the saltus of a function in function theory and to the frontier of a set in topology. We have immediately:

(2.15) (a) $\delta(f) \geqq 0$, *and* $\delta(f) = 0$ *if and only if* $f \in C$,
(b) $\delta(f)$ *is u.s.c.*,
(c) $\delta(-f) = \delta(f)$,
(d) $\delta(\lambda f) = |\lambda| \delta(f)$,

(2.16) $$\begin{matrix} \delta(f \vee g) \\ \delta(f \wedge g) \end{matrix} \leqq \delta(f) \vee \delta(g).$$

(2.17) $$\delta(f) - \delta(g) \leqq \begin{matrix} \delta(f+g) \\ \delta(f-g) \end{matrix} \leqq \delta(f) + \delta(g).$$

Since f and g can be interchanged in (2.17), we have

(2.18) $$|\delta(f) - \delta(g)| \leqq \begin{matrix} \delta(f+g) \\ \delta(f-g) \end{matrix}.$$

We can strengthen one of the inequalities in (2.17):

(2.19) $\delta(f+g) \leqq \delta(f \vee g) + \delta(f \wedge g) \leqq \delta(f) + \delta(g)$.

Setting $g = 0$ in (2.19), and applying (c) of (2.15), we obtain

(2.20) $\delta(f) = \delta(f^+) + \delta(f^-)$.

COROLLARY. $\delta(f) \leqq 2\, u(|f|)$.

Proof. Since $f^+ \geqq 0$, $\delta(f^+) \leqq u(f^+) \leqq u(|f|)$, and similarly for $\delta(f^-)$.

Combining the corollary with (2.17) and (2.14),

(2.21) $\|\delta(f) - \delta(g)\| \leqq 2\|f - g\|$. *Thus the operation* δ *is norm-continuous.*

(2.22) *Given a component e of* **1**, $u(e)$ *and* $l(e)$ *are also components of* **1**.

Now, given a closed ideal I in M, let us denote by $u(I)$ and $l(I)$ the ideals generated by $u(\mathbf{1}_I)$ and $l(\mathbf{1}_I)$ respectively. From the above theorem, $u(I)$ and $l(I)$ are also closed, and $(l(I))' = u(I')$; otherwise stated, if $M = I \oplus J$, then $M = l(I) \oplus u(J)$.

We know [7, (4.3)] that $\bar{C} = M$. In general, given a closed ideal I in M, and setting $D = C \cap I$ (for typographical reasons), we need not have $\bar{D} = I$: for example, $C \cap M_1 = 0$. The precise description of $\bar{D}$ is given by

(2.23) *If* I *is any closed ideal in* M, *then setting* $D = C \cap I$, $\bar{D} = l(I)$ (*hence* $D' = u(I')$).

Proof. We note first that $C \cap I = C \cap l(I)$. For, $f \in C \cap I$, $f \geqq 0$, implies $f \leqq \lambda \mathbf{1}_I$ $(\lambda \geqq 0)$, hence $f \leqq l(\lambda \mathbf{1}_I) = \lambda l(\mathbf{1}_I)$, and thus $f \in l(I)$. It follows we can assume, for simplicity, that $I = l(I)$. Now $\bar{S} = M$, hence $\overline{S_I} = I$. But $\mathbf{1}_I$ is l.s.c., and therefore $S_I = S \cap I$. Thus, to complete the proof, it is enough to show every element of $S \cap I$ is in the closure of $C \cap I$. Since $\mathbf{1}_I$ is l.s.c., an element of $S \cap I$ is the difference of two positive l.s.c. elements *in* I; each of these last is the supremum of elements of $C \cap I$.

Given an ideal I in M, $I^{\perp}$-in-L $= (\bar{I})^{\perp}$-in-L (since $L = \tilde{\Omega}(M)$), and is of course w(L,M)-closed. It is not, in general, w(L,C)-closed. However,

(2.24) *Given an ideal* I *in* M, *the* w(L,C)-*closure of* $I^{\perp}$-*in*-L *is* $(l(\bar{I}))^{\perp}$-*in*-L.

Proof. For simplicity, we omit the designations "in L" and "in M." From [7, (3.3)] and the fact that $M = \tilde{\Omega}(L)$, $I^{\perp\perp} = \bar{I}$. It follows $(I^{\perp})^{\perp}$-in-C $= C \cap \bar{I}$, hence, the w(L,C)-closure of $I^{\perp}$ is $(C \cap \bar{I})^{\perp}$. From (2.23) this last is $(l(\bar{I}))^{\perp}$, and we are through.

We close this section with a resumé of the relations between our operations u, l and the topology of X. Let I be a closed ideal in M and $F = \{x \in X \mid f(x) = 0$ for all $f \in C \cap I\}$. From our discussion in [8, §4], $C(F) = C/C \cap I$, $L(F) = (C \cap I)^{\perp}$-in-L, and $M(F) = (C \cap I)'$-in-M (whence $C(F) = C_{M(F)}$). From our preceding discussion, we also have $L(F) = (l(I))^{\perp}$, $M(F) = u(I')$, and $C(F) = C_{u(I')}$.

3. **The ideal Ra of rare elements.** We will call $f \geqq 0$ *rare* if $l(u(f)) = 0$; more generally, $f \in M$ will be called *rare* if $|f|$ is rare.

(3.1) *Given* $f \in M$, *the following statements are equivalent*:

1° f *is rare*;

2° f^+, f^- *are both rare*;

3° $l(u(f)) = u(l(f)) = 0$;

4° $l(u(f)) \leqq 0 \leqq u(l(f))$.

Proof. We need only show that 4° implies 1°. $u(|f|) = u(f \vee (-f)) = u(f) \vee u(-f)$ (2.4). Applying (2.7) and (2.2), $l(u(|f|)) \leqq u(f) \vee l(u(-f)) = u(f) \vee (-u(l(f))) \leqq u(f) \vee 0$. It follows $l(u(|f|)) \leqq l(u(f) \vee 0) = l(u(f)) \vee 0$ (2.9) $= 0$. Since $l(u(|f|)) \geqq 0$, this gives equality and completes the proof.

(3.2) *The rare elements constitute a norm-closed ideal in* M; *we denote it by* Ra.

Proof. To show Ra is an ideal, it is enough, because of (a) in (2.2), to show that if f and g are two positive rare elements, then so is f + g and so is λf for all $\lambda \geqq 0$. Well, from (2.5), $l(u(f+g)) \leqq l(u(f)+u(g)) \leqq l(u(f))+u(g) = u(g)$; it follows $l(u(f+g)) \leqq l(u(g)) = 0$. That $l(u(\lambda f)) = 0$ follows from (2.2).

It remains to prove Ra is norm-closed. From the corollary to (2.14), $\| l(u(f)) - l(u(g)) \| \leqq \| f - g \|$, $\| u(l(f)) - u(l(g)) \| \leqq \| f - g \|$. This implies that property 3° in (3.1) is retained on passage to the norm-closure of Ra, which completes the proof.

Ra is, in general, not σ-closed. However, it does have the following property (the Category Theorem).

(3.3) *If* $f = \bigvee_{n=1}^{\infty} f_n$, $\{f_n\} \subset Ra_+$, *then* $l(f) = 0$.

Proof. Suppose on the contrary that there exists $g \in C$, $0 < g \leqq f$, and set $\lambda = \| g \|$. Since $g = \bigvee_n (g \wedge u(f_n))$, we can assume, to start with, that the f_n's are u.s.c. and $f \in C$. We will obtain by induction a sequence $\{g_n\} \subset C$ satisfying

(i) $g_1 \leqq g_2 \leqq \cdots \leqq f$,

(ii) $g_n \geqq f_n$, $n = 1, 2, \cdots$,

(iii) $g_n \ngeqq f - (\lambda/2)\mathbf{1}$, $n = 1, 2, \cdots$.

We first remark that $f - (\lambda/2)\mathbf{1} \nleqq 0$, by definition of λ. Since $f - (\lambda/2)\mathbf{1} \in C$, it follows that for any $h \in M$, $l(h) = 0$ implies $h \ngeqq f - (\lambda/2)\mathbf{1}$. We proceed to obtain our sequence. $l(f_1) = 0$, hence by the remark, $f_1 \ngeqq f - (\lambda/2)\mathbf{1}$. But f_1 is the infimum of all the elements of C which lie above it and below f, hence, for at least one of these elements, g_1, we also have $g_1 \ngeqq f - (\lambda/2)\mathbf{1}$. Assume $g_1, \cdots, g_n$ have been chosen to satisfy (i), (ii), (iii).

(iv) $g_n \vee f_{n+1} \ngeqq f - (\lambda/2)\mathbf{1}$.

For, suppose $f - (\lambda/2)\mathbf{1} \leqq g_n \vee f_{n+1} \leqq g_n + f_{n+1}$; then $f - (\lambda/2)\mathbf{1} - g_n \leqq f_{n+1}$, hence $f - (\lambda/2)\mathbf{1} - g_n \leqq 0$ (f_{n+1} is rare), or $g_n \leqq f - (\lambda/2)\mathbf{1}$, contradicting (iii). It follows from (iv), as in the choice of g_1, that there exists $g_{n+1} \in C$ such that $g_n \vee f_{n+1} \leqq g_{n+1} \leqq f$, $g_{n+1} \ngeqq f - (\lambda/2)\mathbf{1}$.

We thus have a sequence $\{g_n\}$ satisfying (i), (ii), (iii). Combining $f = \bigvee_n f_n$ with (i) and (ii) gives us that $g_n \uparrow f$. Applying the Dini theorem, $\lim_{n\to\infty} \| f - g_n \| = 0$, giving us a contradiction with (iii).

Suppose f is l.s.c. and $g \in Ra$; then, for any $h \in C$, $h \geqq f - g$ implies $h \geqq f$. For, $f - h \leqq g$, hence, from (3.1), $f - h \leqq 0$ (since $f - h$ is l.s.c.). Similarly, if the above f is u.s.c., then for any $h \in C$, $h \leqq f + g$ implies $h \leqq f$. Consequently,

(3.4) (a) *If* f *is l.s.c. and* $g \in Ra$, *then* $u(f \pm g) \geqq u(f)$.

(b) *If* f *is u.s.c. and* $g \in Ra$, *then* $l(f \pm g) \leqq l(f)$.

More generally, the conclusions hold if g is any positive element such that $l(g) = 0$.

Combining this with (2.14),

(3.5) *If* $f \in C$ *and* $g \in Ra$, *then* $\|f \pm g\| \geqq \|f\|$.

REMARK. The argument for (3.5) also shows that (3.5) holds if f is a positive l.s.c. element. We will need this later.

We will also need the following two propositions.

(3.6) (a) $f \in Ra$ *implies* $\delta(f) \in Ra$.
(b) $f \in S$ *implies* $\delta(f) \in Ra$.

Proof. (a) follows from the corollary to (2.20). To show (b), it suffices to assume that f is u.s.c. (2.17). Then $u(\delta(f)) = \delta(f) = f - l(f)$, hence $l(u(\delta(f))) = l(f - l(f)) \leqq l(f) - l(f) = 0$, from (2.6), and we are through.

(3.7) *Given a family* $\{g_\alpha\}$ *of l.s.c. elements, if* $f \wedge g_\alpha \in Ra$ *for all* α, *then* $\bigvee_\alpha (f \wedge g_\alpha) \in Ra$.

Proof. Denote $\bigvee_\alpha (f \wedge g_\alpha)$ by g; we show $l(u(g)) \leqq 0 \leqq u(l(g))$, which will give us $g \in Ra$ (3.1). $l(u(f \wedge g_\alpha)) \leqq 0$ for all α, by (3.1), hence $\bigvee_\alpha l(u(f \wedge g_\alpha)) \leqq 0$, hence $u[\bigvee_\alpha l(u(f \wedge g_\alpha))] \leqq 0$. Now apply (2.8). To show $u(l(g)) \geqq 0$, choose an arbitrary α; then $g \geqq f \wedge g_\alpha$, hence $u(l(g)) \geqq u(l(f \wedge g_\alpha)) \geqq 0$ (3.1).

4. **A decomposition of** L. The first consequence of singling out the ideal Ra as an object of interest is the resulting decomposition of L and M. We assume a familiarity with the discussion in [8, §2].

Since Ra is an ideal, its closure $\overline{Ra}$ is also, and we can write $M = \overline{Ra} \oplus Ra'$. Further, $Ra^\perp$-in-L is a closed ideal, since $M = \tilde{\Omega}(L)$, and we have $L = (Ra^\perp\text{-in-}L) \oplus (Ra^\perp\text{-in-}L)'$ — we write it simply $L = Ra^\perp \oplus Ra^{\perp\prime}$. We note also that since $L = \tilde{\Omega}(M)$, $Ra^\perp = (\overline{Ra})^\perp$. Concerning the following theorem, cf. [3, Proposition 1] and [15, §22].

$$(4.1) \qquad \begin{aligned} Ra^\perp &= \tilde{\Omega}(C), \\ Ra^{\perp\prime} &= \tilde{\Omega}(Ra). \end{aligned}$$

Thus, $L = \tilde{\Omega}(C) \oplus \tilde{\Omega}(Ra)$.

Proof. Consider $\mu \in Ra^\perp$, $\mu \geqq 0$, and suppose $f_\alpha \downarrow 0$ in C; we show $\lim_\alpha \mu(f_\alpha) = 0$. If we set $f = \bigwedge_\alpha f_\alpha$ (in M, of course), then clearly $f \in Ra$, hence $\mu(f) = 0$. But $L = \tilde{\Omega}(M)$, and therefore $\lim_\alpha \mu(f_\alpha) = \mu(f) = 0$. Thus, $Ra^\perp \subset \tilde{\Omega}(C)$. Conversely, consider $\mu \in \tilde{\Omega}(C)$, $\mu \geqq 0$, and any element f of Ra_+. Let $\{f_\alpha\}$ be the descending net of all elements of $C \geqq f$. Then $f_\alpha \downarrow 0$ in C, hence $\lim_\alpha \mu(f_\alpha) = 0$. Since $0 \leqq \mu(f) \leqq \mu(f_\alpha)$ for all α, we have $\mu(f) = 0$. Thus $\mu \in Ra^\perp$, and the first equality is proved.

To prove the second equality, we call on the general theorem

(4.2) *Let* E *be a complete vector lattice and* $\tilde{\Omega}(E)$ *separating on* E. *Then for any ideal* I *in* E,

$$\tilde{\Omega}(I) \cap \Omega(E) = I^{\perp\prime}\text{-}in\text{-}\tilde{\Omega}(E).$$

A proof is given in the appendix (I). Applying this to the ideal Ra in M, we have that $Ra^{\perp\prime} = \tilde{\Omega}(Ra) \cap \Omega(M)$. Now Ra has the additional property that it is norm-closed, hence, from standard Banach space theory, that $\Omega(Ra)$ can be identified with $\Omega(M)/(Ra^{\perp}\text{-in-}\Omega(M))$ and therefore with $R^{\perp\prime}\text{-in-}\Omega(M)$. Thus, $\Omega(Ra) \subset \Omega(M)$; a fortiori $\tilde{\Omega}(Ra) \subset \Omega(M)$, and we have $Ra^{\perp\prime} = \tilde{\Omega}(Ra)$.

From (4.1) we have

$$(4.3) \qquad \begin{aligned} Ra' &= \tilde{\Omega}^2(C), \\ \overline{Ra} &= \tilde{\Omega}^2(Ra). \end{aligned}$$

Thus, $M = \tilde{\Omega}^2(C) \oplus \tilde{\Omega}^2(Ra)$.

(4.4) Corollary. *The following two statements are equivalent*:

1° $\tilde{\Omega}(C) = 0$,

2° Ra *is a dense ideal in* M.

An example where $\tilde{\Omega}(C) = 0$ is supplied by taking for X the real interval [0, 1]. More generally, this is true for any compact space X with no isolated points which contains a countable dense set. For, if X satisfies these conditions, then given any $\mu \in L$, there exists a nowhere dense set in X of nonzero μ-measure. We state this in a slightly more general form (cf. [18]).

(4.5) *If* X *has no isolated points, then each* $\mu \in \tilde{\Omega}(C)$ *vanishes on every separable closed subset of* [18, Corollaire 2] X.

As another example where $\tilde{\Omega}(C) = 0$, let N be the set of natural numbers, βN its Stone-Čech compactification, and take for X the complement $\beta N \backslash N$ of N in βN. X has the property that the intersection of a descending sequence of open sets has nonempty interior [4, 6S]. Hence, as Dieudonné points out [3, Lemma 8], not only is $\tilde{\Omega}(C) = 0$, but every element of L has a nowhere dense support.

At the other extreme, let X be the Alexandroff one-point compactification αN of N, and denote the new point by y. Then $\tilde{\Omega}(Ra)$ consists only of the one-dimensional linear subspace Ry generated by y, and $\tilde{\Omega}(C) = l^1(N)$. The case $\tilde{\Omega}(Ra) = 0$ occurs if and only if X is a finite set. We discuss this together with additional examples at the end of this section.

To obtain further insight into the pair of decompositions, $L = \tilde{\Omega}(Ra) \oplus \tilde{\Omega}(C)$, $M = \overline{Ra} \oplus Ra'$, we turn to a different pair of decompositions. In the present paper, G will always denote the open subset of X consisting of all the isolated points, and K will denote the complement of G. Since K is compact, we have the Banach lattice C(K) of continuous functions on K, and its first and second duals L(K), M(K).

It is easy to see that if f is an element of C which vanishes on all of K, then for each $\lambda > 0$, $|f(x)| \geqq \lambda$ for only a finite number of x's. Thus, the subset of C $\{f \in C \mid f(x) = 0 \text{ for all } x \in K\}$ (which is clearly an ideal in C), is precisely $(c_0)(G)$. From the discussion in [8, §4]: $(c_0)(G)$ is norm-closed; $C/(c_0)(G) = C(K)$; $(c_0)(G)^{\perp} = L(K)$; and $(c_0)(G)^{\perp\prime} = l^1(G)$. Thus

$$\text{(4.6)} \qquad \begin{aligned} L &= l^1(G) \oplus L(K) \quad \textit{and (therefore)}, \\ M &= l^\infty(G) \oplus M(K). \end{aligned}$$

We develop the relationship between this pair of decompositions and our preceding pair in the next three theorems. Since $L_0 = l^1(X)$ [6, (4.6)], the decomposition $X = G \cup K$ gives $L_0 = l^1(G) \oplus l^1(K) = l^1(G) \oplus (L(K))_0$, and $M_0 = l^\infty(G) \oplus l^\infty(K) = l^\infty(G) \oplus (M(K))_0$.

$$\text{(4.7)} \qquad \begin{aligned} l^1(G) &= (\tilde{\Omega}(C))_0, \quad (L(K))_0 = (\tilde{\Omega}(Ra))_0. \\ l^\infty(G) &= (Ra')_0, \quad (M(K))_0 = (\overline{Ra})_0. \end{aligned}$$

Proof. Each $x \in G$ is clearly a continuous linear functional on C: $x \in \tilde{\Omega}(C)$. This gives $G \subset \tilde{\Omega}(C)$, hence, $l^1(G) \subset \tilde{\Omega}(C)$. On the other hand, consider $x \in K$ and let g be the element of M_0 such that $g(x) = 1$, $g(y) = 0$ for all $y \in X$, $y \neq x$. Since x is not isolated, $g \in Ra$, hence $x \in \tilde{\Omega}(Ra)$. Thus, $K \subset \tilde{\Omega}(Ra)$, whence. $(L(K))_0 \subset \tilde{\Omega}(Ra)$. The decomposition of L_0 above now gives the first two equalities of the theorem. The last two equalities follow from the first.

From (4.6) and (4.7) we obtain

$$\text{(4.8)} \qquad \begin{aligned} L(K) &= \tilde{\Omega}(Ra) \oplus (\tilde{\Omega}(C))_1, \\ M(K) &= \overline{Ra} \oplus (Ra')_1. \end{aligned}$$

Less trivial (cf. the discussion following (2.22)) is

(4.9) *The following equivalent properties hold*:

(a) $L(K)$ *is the* $w(L,C)$-*closure of* $\tilde{\Omega}(Ra)$;

(b) $M(K) = u(\overline{Ra})$;

(c) $l^\infty(G) = l(Ra')$.

Proof. That (b) and (c) are equivalent follows from $l^\infty(G) \oplus M(K) = Ra' \oplus \overline{Ra}$ and the discussion following (2.22). That (a) is equivalent to them follows from (2.24). It is thus enough to prove (c), and for this it is enough, from (2.23) to show

(i) $C \cap Ra' = (c_0)(G)$.

Suppose $f \in C \cap Ra'$, $f \geqq 0$; we show f vanishes on all of K. Consider $x \in K$ and let $f(x) = \lambda \geqq 0$. Then $\lambda g \leqq f$, where g is the element of M_0 defined in the proof of (4.7). Since $f \in Ra'$ and $g \in Ra$, we must have $\lambda = 0$. This proves (i), and with it (4.9).

We consider the question: which of the four summands $(\tilde{\Omega}(C))_0$, $(\tilde{\Omega}(C))_1$, $(\tilde{\Omega}(Ra))_0$, and $(\tilde{\Omega}(Ra))_1$ (and their dual summands in M) can reduce to 0? As we saw earlier, we may have $\tilde{\Omega}(C) = 0$, hence a fortiori, $(\tilde{\Omega}(C))_0 = (\tilde{\Omega}(C))_1 = 0$. As an example where $(\tilde{\Omega}(C))_1 = 0$ while $(\tilde{\Omega}(C))_0 \neq 0$, let X be any space for which K has empty interior—αN or βN will do. Then $C \cap M(K) = 0$, hence $\mathbf{1}_{M(K)} \in Ra$, hence $M(K) = Ra$, and from (4.8), $(Ra')_1 = 0$, whence of course $(\tilde{\Omega}(C))_1 = 0$.

We may also have $(\tilde{\Omega}(C))_0 = 0$ while $(\tilde{\Omega}(C))_1 \neq 0$: Let μ be the Lebesgue measure on the linear interval $[0,1]$, and take for C the space $\mathscr{L}^\infty(\mu)$, that is, let X be the Kakutani-Stone space of $\mathscr{L}^\infty(\mu)$. As we know [7, (5.4)], $\tilde{\Omega}(C) = \mathscr{L}^1(\mu)$, hence is even separating on C. However, $(\tilde{\Omega}(C))_0 = 0$. For, if X contained an isolated point, $\tilde{\Omega}(C)$ would contain an element not expressible as the sum of two disjoint elements; but $\mathscr{L}^1(\mu)$ contains no such element.

Another example is furnished by taking for C the entire space $(M([0,1]))_1$, or more generally, any $M_1 \neq 0$. In Dixmier's terminology, the X in the paragraph above is of "genre denombrable," while that obtained from $(M([0,1]))_1$ is not.

Turning to $\tilde{\Omega}(Ra)$, $(\tilde{\Omega}(Ra))_0 = 0$ if and only if X is a finite set, since an infinite compact set contains at least one nonisolated point. Since for X finite, $\tilde{\Omega}(Ra) = 0$, we have that $(\tilde{\Omega}(Ra))_0 = 0$ if and only if $\tilde{\Omega}(Ra) = 0$. We can have $(\tilde{\Omega}(Ra))_1 = 0$ for infinite X: αN furnishes such an example.

5. **On** $\tilde{\Omega}(C)$. Nakano [12] and Dixmier [3] have studied $\tilde{\Omega}(C)$ for the case where C is (vector lattice) complete. Actually, Nakano considers $\tilde{\Omega}(E)$, E any complete vector lattice, while Dixmier, like ourselves, confines himself to $\tilde{\Omega}(C)$ (he calls its elements the *normal* measures on X). However, many of the properties of $\tilde{\Omega}(C)$ are not dependent on the completeness of C. We discuss some of these in the present section, and take up the case where C is complete later (§11).

We recall some notation and definitions from [6, §12]. Given $\mu \in L$, then by our general notation, L_μ is the closed ideal generated by μ. We denote its dual $L_\mu^{\perp}{}'$-in-M by M_μ. L_μ can be identified with $\mathscr{L}^1(\mu)$ and M_μ with $\mathscr{L}^\infty(\mu)$. The ideal in M generated by $L_\mu^\perp \cap U$ $(= M_\mu' \cap U)$ is denoted by $N(\mu)$, and its elements are called *μ-negligible*. And $U + N(\mu)$ is denoted by $\mathscr{M}(\mu)$, and its elements called *μ-integrable*. From [6, (12.7)], $f \in N(\mu)$ implies $f_* \in N(\mu)$ and $f^* \in N(\mu)$. In general it does not imply $l(f) \in N(\mu)$ or $u(f) \in N(\mu)$ (note that $l(f) \leqq f_* \leqq f^* \leqq u(f)$). However,

(5.1) *If* $\mu \in \tilde{\Omega}(C)$, *then* $f \in N(\mu)$ *implies* $l(f) \in N(\mu)$, $u(f) \in N(\mu)$.

Proof. From (2.12) it is enough to consider $f \geqq 0$ and prove $u(f) \in N(\mu)$. Assume first that $f \in L_\mu^\perp \cap U$. Let $\{f_\alpha\}$ be the descending net of all elements of $S \geqq f$; by definition of U, $f_\alpha \downarrow f$. It follows $\inf_\alpha \mu(f_\alpha) = \mu(f) = 0$. Since $\mu \in Ra^\perp$, $\mu(u(f_\alpha)) = \mu(f_\alpha)$ for all α (3.6). Thus $\inf_\alpha \mu(u(f_\alpha)) = 0$, and the inequality $u(f) \leqq u(f_\alpha)$ for all α gives us $\mu(u(f)) = 0$. Now, let f be any positive element of

$N(\mu)$. $f \leqq g$ for some $g \in L_\mu^\perp \cap U$. Since $u(f) \leqq u(g) \in N(\mu)$, it follows $u(f) \in N(\mu)$ also.

(5.2) [3, Proposition 2]. *If* $\mu \in \tilde{\Omega}(C)$, *then for every* f *which is* μ-*integrable,*

$$(l(f))_\mu = f_\mu = (u(f))_\mu.$$

Proof. We note first that from (3.6), the theorem holds for all $f \in S$. Next, consider $f \in U$. $f \leqq u(f)$, hence, $f_\mu \leqq (u(f))_\mu$; we show the opposite inequality. Again, let $\{f_\alpha\}$ be the set of all elements of S which are $\geqq f$. Then, on the one hand (i) $f_\mu = \bigwedge_\alpha (f_\alpha)_\mu$, and on the other, (ii) $u(f) = \bigwedge_\alpha u(f_\alpha)$. It follows $(u(f))_\mu = \bigwedge_\alpha (\mu(f_\alpha))_\mu = \bigwedge_\alpha (f_\alpha)_\mu = f_\mu$. We thus have the second equality in the statement of the theorem for $f \in U$; the first is shown similarly. Finally, consider $f = g + h, g \in U, h \in N(\mu)$. Again, we need only show $(u(f))_\mu \leqq f_\mu$. $u(f) \leqq u(g) + u(h)$ (2.5), hence, $(u(f))_\mu \leqq (u(g))_\mu + (u(h))_\mu$. Applying $(u(h))_\mu = h_\mu = 0$ (5.1) and the fact that the theorem holds for g, we obtain $(u(f))_\mu \leqq (u(g))_\mu = g_\mu = f_\mu$, and we are through.

Corollary. *If* $\mu \in \tilde{\Omega}(C)$, *then for each* $h \in M_\mu$, $h \geqq 0$, *we have* $(u(h))_\mu = h$.

Proof. There exists $f \in Ba$ such that $f_\mu = h$ (cf. [6, (12.4)]), hence by (5.2), $(u(f))_\mu = h$. Since $h \geqq 0$, $h \leqq u(f)$, therefore $h \leqq u(h) \leqq u(f)$, and we have the desired result.

We have another consequence of (5.2). In the proof of the corollary, we recalled that $M_\mu = (Ba)_\mu$. This is true for any $\mu \in L$. For $\mu \in \tilde{\Omega}(C)$, we can say more.

(5.3) *Given* $\mu \in \tilde{\Omega}(C)$, $M_\mu = (Ba \cap S)_\mu$. *More precisely, each* $h \in M_\mu$ *can be written* $h = f_\mu$ *for some* $f \in Ba$ *which is u.s.c.* (*and also* $h = f_\mu$ *for some* $f \in Ba$ *which is l.s.c.*).

Proof. For simplicity, we can assume $\mu \geqq 0$. Consider $h \in M_\mu$ and choose g u.s.c. such that $g_\mu = h$ (5.2). Let $A = \{f \in C \mid f \geqq g\}$; then $g = \bigwedge A$, hence $\mu(g) = \inf_{f \in A} \mu(f)$. Choose $\{f_n\} \subset A$ such that $\mu(f_n) \leqq \mu(g) + 1/n$ $(n = 1, 2, \cdots)$, and let $f = \bigwedge_n f_n$. f is a u.s.c. element in Ba; we show $f_\mu = h$. $\mu(f) = \inf_n \mu(f_n) = \mu(g)$, hence $\mu(f-g) = 0$; since $f - g \geqq 0$, it follows $(f-g)_\mu = 0$, whence $f_\mu - g_\mu = 0$ and thus $f_\mu = g_\mu = h$.

(5.4) (Cf. [3, Proposition 5].) *If* $\tilde{\Omega}(C)$ *is separating on* C, *then*

$$\bigcap_{\mu \in \tilde{\Omega}(C)} N(\mu) = Ra.$$

Proof. Consider $\mu \in \tilde{\Omega}(C)$, and let f be any element of Ra, $f \geqq 0$. $u(f) \in Ra$, hence $u(f) \in L_\mu^\perp$, hence, $u(f) \in N(\mu)$, hence $f \in N(\mu)$. Thus, $Ra \subset N(\mu)$ for every $\mu \in \tilde{\Omega}(C)$. Conversely, suppose $f \geqq 0$, $f \notin Ra$. Then, there exists $g \in C$ such that $0 < g \leqq u(f)$, and since $\tilde{\Omega}(C)$ is separating on C, $\mu(g) > 0$ for some $\mu \in \tilde{\Omega}(C)$, $\mu \geqq 0$. This gives $\mu(u(f)) > 0$, hence $u(f) \notin N(\mu)$, hence, from (5.1), $f \notin N(\mu)$.

Since $N(\mu)$ is σ-closed for every μ [6, (12.6)], we obtain the

COROLLARY. *If* $\tilde{\Omega}(C)$ *is separating on* C, *then* Ra *is* σ*-closed.*

(5.5) [3, Proposition 6] *If* $\tilde{\Omega}(C)$ *is separating on* C, *then*

$$\bigcap_{\mu \in \tilde{\Omega}(C)} \mathscr{M}(\mu) = \{f \in M \,|\, \delta(f) \in Ra\} = S + Ra.$$

Proof. If $f \in \mathscr{M}(\mu)$, then $(\delta(f))_\mu = (u(f) - l(f))_\mu = (u(f))_\mu - (l(f))_\mu = 0$ (5.2); since $\delta(f)$ is u.s.c., it follows $\delta(f) \in N(\mu)$. Combining this with (5.4), if $f \in \mathscr{M}(\mu)$ for all $\mu \in \tilde{\Omega}(C)$, then $\delta(f) \in Ra$. Conversely, if $\delta(f) \in Ra$, then for every $\mu \in \tilde{\Omega}(C)$, $\delta(f) \in N(\mu)$, hence $f - l(f) \in N(\mu)$, hence $f \in S + N(\mu) \subset \mathscr{M}(\mu)$. The second equality in the theorem is always true; it is proved in (8.3).

6. **Homomorphisms of** C. In this section we develop some properties of linear transformations of vector lattices, and in particular of C, which we will need. It will be assumed below, without explicit mention, that $\Omega(E)$ and $\Omega(F)$ are separating on E and F respectively.

A linear transformation $T: E \to F$ of one vector lattice into another will be called a *positive transformation* if $TE_+ \subset F_+$, or equivalently, if it is order-preserving. For later use, we note the property $|Ta| \leqq T|a|$ for every $a \in E$.

The definition, $(T^t\psi)(a) = \psi(Ta)$ for all $\psi \in \Omega(F)$ and $a \in E$, defines a linear transformation $T^t: \Omega(F) \to \Omega(E)$—the *transpose* of T—which is also positive. From standard vector space theory, T is continuous with respect to the topologies $w(E, \Omega(E))$, $w(F, \Omega(F))$, and T^t is continuous with respect to $w(\Omega(F), F)$, $w(\Omega(E), E)$. We are more interested in order-continuity. T will be called *continuous* if $a_\alpha \to a$ in E implies $Ta_\alpha \to Ta$ in F.

(6.1) *Given a positive transformation* $T: E \to F$, T^t *is continuous.*

Proof. It is enough to show that $\psi_\alpha \downarrow 0$ in $\Omega(F)$ implies $T^t\psi_\alpha \downarrow 0$ in $\Omega(E)$. Since T^t is order-preserving, $T^t\psi_\alpha \downarrow$ (that is, $\alpha \prec \beta$ implies $T^t\psi_\beta \leqq T^t\psi_\alpha$) and $T^t\psi_\alpha \geqq 0$ for all α; thus, we need only show $\inf_\alpha (T^t\psi_\alpha)(a) = 0$ for all $a \in E_+$. Since $\inf_\alpha (T^t\psi_\alpha)(a) = \inf_\alpha \psi_\alpha(Ta)$, we are through.

(6.2) *Given a positive transformation* $T: E \to F$. *If* T *is continuous, then* $T^t\tilde{\Omega}(F) \subset \tilde{\Omega}(E)$. *And when* $\tilde{\Omega}(F)$ *is separating on* F, *the converse holds.*

Proof. Assume T is continuous, and consider $\psi \in \tilde{\Omega}(F)$. Then if $a_\alpha \to a$ in E, $(T^t\psi)(a_\alpha) = \psi(Ta_\alpha) \to \psi(Ta) = (T^t\psi)(a)$. Thus, $T^t\psi \in \tilde{\Omega}(E)$. Now, let $\tilde{\Omega}(F)$ be separating on F and assume $T^t\tilde{\Omega}(F) \subset \tilde{\Omega}(E)$. Suppose $a_\alpha \downarrow 0$ in E; we show $\bigwedge_\alpha T_\alpha a = 0$. Consider $0 \leqq b \leqq Ta_\alpha$ for all α, and let ψ be any element of $\tilde{\Omega}(F)$, $\psi \geqq 0$. Then, $0 \leqq \psi(b) \leqq \psi(Ta_\alpha) = (T^t\psi)(a_\alpha)$ for all α; hence, $0 \leqq \psi(b) \leqq \inf_\alpha (T^t\psi)(a_\alpha) = 0$. Since ψ was an arbitrary positive element of $\tilde{\Omega}(F)$, it follows $b = 0$. Thus, $Ta_\alpha \downarrow 0$ in F, and we are through.

Starting with a positive transformation $T: E \to F$, we have $T^t: \Omega(F) \to \Omega(E)$, and this in turn defines $T^{tt}: \Omega^2(E) \to \Omega^2(F)$. E is a subspace of $\Omega^2(E)$ (in fact, of $\tilde{\Omega}(\Omega(E))$), and on E, T^{tt} is identical with T. (6.1) gives us

(6.3) *Given a positive transformation* T, T^{tt} *is a continuous positive transformation.*

A linear transformation $h:E\rightarrow F$ will be called a *homomorphism* if it preserves the operations $\vee$ and $\wedge$, that is, for every $a,b\in E$, $h(a\vee b)=(ha)\vee(hb)$-in-F, and similarly for $h(a\wedge b)$. A homomorphism is positive, so the preceding discussion applies to it. Given a homomorphism h, hE is a sub-vector-lattice of F and $h^{-1}0$ is an ideal of E. Also, given an ideal I of E, the canonical mapping $\pi:E\rightarrow E/I$ of E onto the quotient vector lattice is a homomorphism.

The following generalizes [7, (2.4)].

(6.4) *If* $h:E\rightarrow F$ *is a homomorphism, then* $h^t\Omega(F)$ *is an ideal in* $\Omega(E)$.

Proof. Set $I=h^{-1}0$. $h^t\Omega(F)\subset I^\perp$, hence, it is enough to show that it is an ideal in $I^\perp$. Now, $I^\perp$ is isomorphic with $\Omega(E/I)$, and E/I is isomorphic with hE; thus, $I^\perp$ is isomorphic with $\Omega(hE)$. Moreover, h^t can be identified with the mapping $\pi:\Omega(F)\rightarrow\Omega(hE)$ defined by $\pi\psi=\psi|hE$. Thus, the theorem is reduced to [7, (2.4)], and we are through.

(6.5) *If* $h:E\rightarrow F$ *is a homomorphism, then* $h^{tt}:\Omega^2(E)\rightarrow\Omega^2(F)$ *is also a homomorphism (and continuous* (6.3)).

Proof. It is enough to show that for every $a\in\Omega^2(E)$, $h^{tt}a^+=(h^{tt}a)^+$. Since h^{tt} preserves order, $h^{tt}a^+\geqq h^{tt}a$, hence $h^{tt}a^+\geqq(h^{tt}a^+)$. We prove the opposite inequality by showing $\psi(h^{tt}a^+)\leqq\psi((h^{tt}a)^+)$ for all $\psi\in\Omega(F)$, $\psi\geqq 0$. Now

$$\psi(h^{tt}a^+)=(h^t\psi)(a^+)=\sup_{0\leqq\phi\leqq h^t\psi}\phi(a),$$

while

$$\psi((h^{tt}a)^+)=\sup_{0\leqq\rho\leqq\psi}\rho(h^{tt}a)=\sup_{0\leqq\rho\leqq\psi}(h^t\rho)(a).$$

Thus the problem reduces to showing that if $0\leqq\phi\leqq h^t\psi$, then $\phi=h^t\rho$ for some ρ satisfying $0\leqq\rho\leqq\psi$. This follows from the proof of (6.4) and the Lemma of [7,(2.4)].

Now let X and Y be compact spaces. If we have a homomorphism $h:C(X)\rightarrow C(Y)$, then from (6.5), $h^{tt}:M(X)\rightarrow M(Y)$ is a continuous homomorphism. The converse is also true provided C(X) is carried into C(Y):

(6.6.) *Given a homomorphism* $H:M(X)\rightarrow M(Y)$, *the following two statements are equivalent*:

1° $H=h^{tt}$ *for some homomorphism* $h:C(X)\rightarrow C(Y)$;

2° *H is continuous and carries* C(X) *into* C(Y).

Proof. Assume 2° holds, and set $h=H|C(X)$; we show $H=h^{tt}$. It is enough to show $\nu(Hf)=\nu(h^{tt}f)$ for all $\nu\in L(Y)$ and $f\in M(X)$. Writing the equality in the form $(H^t\nu)(f)=(h^t\nu)(f)$, we see we need only prove $H^t\nu=h^t\nu$ for all $\nu\in L(Y)$.

Now from (6.2), $H^t L(Y) \subset L(X)$, hence $H^t\nu$ is completely determined by its values on C(X) (and of course $h^t\nu$ is also). Thus, the problem reduces to showing $(H^t\nu)(f) = (h^t\nu)(f)$ for all $\nu \in L(Y)$ and $f \in C(X)$. Writing the equality in the form $\nu(Hf) = \nu(hf)$, and applying the definition of h at the beginning of the proof, we are through.

REMARK 1. Thus given any homomorphism h of C(X) into C(Y), there is one and *only one* extension of h to a continuous homomorphism of M(X) into M(Y), viz. h^{tt}. This uniqueness agrees with the fact that $M = \overline{C}$.

REMARK 2. Note that a homomorphism of C(X) is automatically norm-continuous, since $|f| \leqq \lambda \mathbf{1}$ implies $|hf| \leqq \lambda h\mathbf{1}$.

The continuity of h^{tt} also gives us

(6.7) *Let* $h: C(X) \to C(Y)$ *be a homomorphism. Then*:

(a) *If* $f \in M(X)$ *is u.s.c.*, $h^{tt}f$ *is u.s.c.*; *and similarly for l.s.c. Consequently*, $h^{tt}S(X) \subset S(Y)$.

(b) $h^{tt}Ba(X) \subset Ba(Y)$,

(c) $h^{tt}Bo(X) \subset Bo(Y)$,

(d) $h^{tt}U(X) \subset U(Y)$.

A corollary of (a) is

(6.8) *Let* $h: C(X) \to C(Y)$ *be a homomorphism. Then for every* $f \in M$,

$$h^{tt}u(f) \geqq u(h^{tt}f), \qquad h^{tt}l(f) \leqq l(h^{tt}f).$$

REMARK. The converse of (6.8), or equivalently of (a) of (6.7) also holds: if a homomorphism $H: M(X) \to M(Y)$ is continuous and satisfies the conclusion of either (6.8) or (a) of (6.7), then $H = h^{tt}$ for some homomorphism $h: C(X) \to C(Y)$. For then H carries C(X) onto C(Y) (cf. (6.6)).

A continuous mapping $q: Y \to X$ determines a homomorphism $h: C(X) \to C(Y)$ by $(hf)(y) = f(qy)$ for all $f \in C(X)$, $y \in Y$ (here of course the continuity of q is meant with respect to the topologies of Y and X). It seems natural to call h the *transpose* q^t of q, and we will do so. In addition to being a homomorphism, q^t has the property that $q^t\mathbf{1} = \mathbf{1}$ (i.e., $q^t\mathbf{1}(X) = \mathbf{1}(Y)$). Stone has shown [13] that conversely, a homomorphism of C(X) into C(Y) which carries $\mathbf{1}$ into $\mathbf{1}$ is the transpose of a continuous mapping of Y onto X; otherwise stated, that 1° and 2° in the following theorem are equivalent.

(6.9) *Given a mapping* $h: C(X) \to C(Y)$, *the following statements are equivalent*:

1° h *is a homomorphism satisfying* $h\mathbf{1} = \mathbf{1}$;

2° $h = q^t$, *where* q *is a continuous mapping of* Y *into* X;

3° *the mapping* $h^t: R^{C(Y)} \to R^{C(X)}$, *defined by* $(h^t\psi)(f) = \psi(hf)$ *for all* $\psi \in R^{C(Y)}$, $f \in C(X)$, *carries* Y *into* X.

If one, hence all, of these hold, then clearly $q = h^t|Y$. *Moreover,* h *is one-one if and only if* q *is onto.*

We prove 3° is equivalent to the first two statements. Assume 1° holds and consider $y \in Y$. y is a homomorphism of C(Y) into $R: y(h \wedge k) = \min(y(h), y(k))$ for all $h, k \in C(Y)$. We show $h^t y$ has the same property with respect to C(X). Given $f, g \in C(X)$, $(h^t y)(f \wedge g) = y(h(f \wedge g)) = y(hf \wedge hg) = \min(y(hf), y(hg)) = \min((h^t y(f), (h^t y)(g))$. Thus $h^t y \in X$ or $h^t y = 0$. Since $(h^t y)(\mathbf{1}) = y(h\mathbf{1}) = y(\mathbf{1}) = 1$, the latter possibility is eliminated and we have 3°. Now assume 3° holds, and denote $h^t|Y$ by q. Define $q^t: C(X) \to R^Y$ by $(q^t f)(y) = f(qy)$ for all $f \in C(X)$, $y \in Y$. But then, clearly $q^t f = hf$ for all $f \in C(X)$. This gives us first of all that q^t carries C(X) into C(Y), whence q is continuous, and then that 2° holds. The final statement in the theorem follows immediately from the fact that qY is compact, hence, closed in X.

REMARK. h (i.e., q^t) in the above theorem and, more extensively, h^{tt} corresponds to the operation q^{-1} of set theory (of [15, §11]). Thus, to the two standard conditions for continuity of q: (a) for every closed set Z of X, $q^{-1}Z$ is closed, and (b) for every set Z of X, $Cl(q^{-1}Z) \subset q^{-1}Cl(Z)$, where "Cl" means topological closure, we have (a′) for every u.s.c. element f of M, $h^{tt}f$ is u.s.c., and (b′) for every $f \in M$, $u(h^{tt}f) \leqq h^{tt}u(f)$.

In our discussion of a homomorphism $h: C(X) \to C(Y)$ we have had no continuity assumption on h, that is, we have not assumed that $f_\alpha \downarrow f$ in C(X) implies $hf_\alpha \downarrow hf$ in C(Y). For the following theorem, cf. [15, Theorem 22.3].

(6.10) *Given a homomorphism* $h: C(X) \to C(Y)$, *the following are equivalent*:

1° h *is continuous*;

2° $h^{tt}Ra(X) \subset Ra(Y)$.

Proof. Assume h is continuous, and consider $f \in Ra(X)$, $f \geqq 0$. Let $B = \{g \in C(Y) | g \geqq h^{tt}f\}$; we show $\bigwedge B\text{-in-}C(Y) = 0$, whence it will follow $h^{tt}f \in Ra(Y)$. Let $A = \{k \in C(X) | k \geqq f\}$. Then (i) $\bigwedge A\text{-in-}C(X) = 0$ and (ii) $hA \subset B$. It follows from (i) and the continuity of h that $\bigwedge hA\text{-in-}C(Y) = 0$. But from (ii), $\bigwedge B\text{-in-}C(Y) \leqq \bigwedge A\text{-in-}C(Y)$.

Now assume 2° holds, and suppose $f_\alpha \downarrow 0$ in C(X). Let $f = \bigwedge f_\alpha$; then $hf_\alpha = h^{tt}f_\alpha \downarrow h^{tt}f$ (since h^{tt} is continuous). But $f \in Ra(X)$, hence $h^{tt}f \in Ra(Y)$, hence $\bigwedge_\alpha hf_\alpha\text{-in-}C(Y) = 0$.

COROLLARY. *If* $h: C(X) \to C(Y)$ *is a continuous homomorphism, then* $h^{tt}\overline{R}a(X) \subset \overline{R}a(Y)$.

An *isomorphism* $h: E \to F$ of one vector lattice into another is a homomorphism which is one-one (into). It follows from standard vector space theory that $h^t\Omega(F)$ is $w(\Omega(E), E)$-dense in $\Omega(E)$, and this in turn gives us that h^{tt} is also one-one. Combining this with (6.5), (6.11) *If* $h:E\to F$ *is an isomorphism of* E *into* F, *then* $h^{tt}:\Omega^2(E)\to\Omega^2(F)$ *is a continuous isomorphism.*

7. Relations between C and the decompositions of M and L. We showed in (i) of (4.9) that $C\cap Ra'=(c_0)(G)$. We can say nothing particular about $C\cap\overline{Ra}$, since we can have on the one hand, $C\cap\overline{Ra}=0$ ($X=\alpha N$), and on the other hand, $C\cap\overline{Ra}=C$ ($X=[0,1]$). What about the intersections of C with $(Ra')_0$, $(Ra')_1$, $(\overline{Ra})_0$, $(\overline{Ra})_1$?

(7.1)(a) $C\cap(Ra')_0=(c_0)(G)$;
(b) $C\cap(Ra')_1=C\cap(\overline{Ra})_1=C\cap(\overline{Ra})_0=0$.

Proof. (a) follows from the opening statement above, which actually gives us the property
(i) $C\cap Ra'\subset(Ra')_0$.
Two of the equalities in (b) follow from the fact that $C\cap M_1=0$. It remains to show $C\cap(\overline{Ra})_0=0$. Since $(\overline{Ra})_0=(M(K))_0$, we can, for simplicity, assume $X=K$, that is, X has no isolated points. Then what we have to show is that $C\cap M_0=0$.

LEMMA. *If* X *has no isolated points,* L_1 *is separating on* C.

This is proved by Loomis in [10]. More exactly, Loomis shows there exists a nontrivial diffuse (regular) measure on X, but the lemma follows easily from this. That $C\cap M_0=0$ in turn follows immediately from the lemma.

We turn to the projections of C in the various components of M. We first note the general theorem.

(7.2) *If* $M=I\oplus J$, *then*
(a) C_I *is norm-closed*;
(b) $\overline{C}_I=I$ (*we recall that* $\overline{C}_I$ *denotes the closure of* C_I);
(c) *if* $C\cap J=0$, *the projection mapping* $C\to C_I$ *is a norm-preserving isomorphism.*

This is not hard to show, and we omit the proof.

COROLLARY. $C_{Ra'}$ *is norm-closed, and* $\overline{C}_{Ra'}=Ra'$. *Similarly,* $C_{\overline{Ra}}$ *is norm-closed, and* $\overline{C}_{\overline{Ra}}=\overline{Ra}$.

Actually, with respect to Ra′, we can say more.

(7.2a) *Every element of* Ra′ *is the projection of a u.s.c. element and also of an l.s.c. element. Thus, every element of* Ra′ *is the supremum of a subset of* $C_{Ra'}$ *and the infimum of a subset of* $C_{Ra'}$.

REMARK. The last says Ra′ is the cut-completion of $C_{Ra'}$ (cf. §9).

Proof of (7.2a). Denote the projection $M\to Ra'$ by p, and consider pS.

From (3.6) each element of pS is the image of a u.s.c. element and of an l.s.c element. We show pS is closed in Ra′, hence from the corollary, $pS = Ra'$. It is enough to show that if $0 \leqq f = \bigwedge A$, $A \subset pS$, then $f \in pS$. Let B be the set of all positive u.s.c. elements carried into A by p; then $A = pB$. Denoting $\bigwedge B$ by g, we have $pg = \bigwedge pB = \bigwedge A = f$, and since g is u.s.c., we are through.

$C_{Ra'}$ itself has an additional property.

(7.3) *Given* $A \subset C_{Ra'}$, *the following are equivalent*:
1° $0 = \bigwedge A$;
2° $0 = \bigwedge A$-*in*-$C_{Ra'}$;
3° $A = B_{Ra'}$, *where* $B \subset C$ *and* $0 = \bigwedge B$-*in*-C.

Proof. 1° of course implies 2°. Assume 2° holds, let $B = \{g \in C_+ \mid pg \in A\}$ (p, the projection $M \to Ra'$), and denote $\bigwedge B$ by k. We show $k \in Ra$, which will give 3°. Consider $h \in C$, $0 \leqq h \leqq k$. Then $0 \leqq ph \leqq pk = \bigwedge pB = \bigwedge A$, hence from 2°, $ph = 0$. It follows $g - h \in B$ for all $g \in B$, whence $\bigwedge_{g \in B} g = \bigwedge_{g \in B}(g - h)$ $= \bigwedge_{g \in B} g - h$, and therefore $h = 0$. We thus have that $l(k) = 0$; since k is u.s.c., it follows $k \in Ra$. Finally, assume 3° holds, and again let $k = \bigwedge B \in Ra$. Then $0 = pk = \bigwedge A$, and we have 1°.

The implication 3° implies 1° gives us the

Corollary. *The projection* $C \to Ra'$ *is a continuous homomorphism.*

(7.4) $\tilde{\Omega}(C) = \tilde{\Omega}(C_{Ra'})$.

By the equality, we mean that $\tilde{\Omega}(C)$ (which is also $\tilde{\Omega}(Ra')$) is isomorphic with $\tilde{\Omega}(C_{Ra'})$ and that the multiplication between $\tilde{\Omega}(C)$ and $C_{Ra'}$ induced by the latter's imbedding in Ra′ gives the isomorphism.

We proceed to prove the theorem. Specifically, denoting the identity mapping $C_{Ra'} \to Ra'$ by i, we show that i^t restricted to $\tilde{\Omega}(Ra')$ is an isomorphism with $\tilde{\Omega}(C_{Ra'})$. In this proof p will denote the restricted projection $p: C \to C_{Ra'}$. From (7.3), p and i are continuous homomorphisms; hence, applying (6.2), i^t carries $\tilde{\Omega}(Ra')$ into $\tilde{\Omega}(C_{Ra'})$ and p^t carries $\tilde{\Omega}(C_{Ra'})$ into $\tilde{\Omega}(C)$. We thus have

$$C \xrightarrow{p} C_{Ra'} \xrightarrow{i} Ra',$$

$$\tilde{\Omega}(C) \xleftarrow{p^t} \tilde{\Omega}(C_{Ra'}) \xleftarrow{i^t} \tilde{\Omega}(Ra') = \tilde{\Omega}(C).$$

Now $p^t \circ i^t$ is the identity mapping on $\tilde{\Omega}(C)$. For, given $\mu \in \tilde{\Omega}(C)$, then for any $f \in C$, $\mu(f) = \mu(f_{Ra'}) + \mu(f_{\overline{Ra}}) = \mu(f_{Ra'}) = \mu(i \circ p^t f) = (p \circ i^t \mu)(f)$; since f was arbitrary, we have $\mu = p^t \circ i^t \mu$. Since p^t is one-one (because p is onto), it follows i^t is onto, and is therefore the desired isomorphism.

We turn to $C_{\overline{Ra}}$. The inverse image of 0 under the projection $C \to C_{\overline{Ra}}$ is $C \cap Ra'$. From (7.1), this is identical with $(c_0)(G)$, hence, from the discussion in §4 ($\cong$ means isomorphic), (7.5) $C_{\bar{R}_a} \cong C_{u(\bar{R}_a)} \cong C(K)$.

This gives

(7.6) (a) $\Omega(C_{\bar{R}_a}) \cong L(K) = \tilde{\Omega}(Ra) \oplus (\tilde{\Omega}(C))_1$,
(b) $\tilde{\Omega}(C_{\bar{R}_a}) \cong \tilde{\Omega}(C(K))$.

The best we can state analogous to (7.4) is

(7.7) $(\tilde{\Omega}(C))_1 \subset \tilde{\Omega}(C(K))$, *hence from* (b) *above*, $\tilde{\Omega}(C_{\bar{R}_a})$ *contains an isomorphic image of* $(\tilde{\Omega}(C))_1$.

This follows from the easily shown property: if $f_\alpha \downarrow 0$ in C(K), there exists (via the Tietze Extension Theorem) a net $g_\beta \downarrow 0$ in C whose restriction to K is a subnet of $\{f_\alpha\}$.

We need not have equality in (7.7): if $X = \alpha N$, K consists of a single point y, whence $C(K) \cong R$ and $y \in \tilde{\Omega}(C(K))$, while $(\tilde{\Omega}(C))_1 = 0$.

8. **The subspaces** C + Ra **and** S + Ra. Since C + Ra is a direct sum, we will write it $C \oplus Ra$. If E_1, E_2 are two sub-vector-lattices of a vector lattice, their sum $E_1 + E_2$ need not also be one. However.

(8.0) *If* F *is a sub-vector-lattice of a vector lattice* E, *and* I *is an ideal in* E, *then*

(a) $F + I$ *is a sub-vector-lattice of* E;
(b) *every* $a \in (F+I)_+$ *can be written* $a = b + c$, *where* $b \in F_+$ *and* $c \in I$.

Proof. To prove (a), we show $(a+b)^+ \in F+I$ for all $a \in F$, $b \in I$. $(a+b)^+ = (a^+ - a^- + b^+ - b^-) \vee 0 = (a^+ + b^+) \vee (a^- + b^-) - (a^- + b^-)$. Thus, it is enough to show that if a, c are in F_+ and b, d are in I_+, then $(a+b) \vee (c+d) \in F + I$. Well, $(a+b) \vee (c+d) \leqq a \vee c + b \vee d$, hence $0 \leqq (a+b) \vee (c+d) - a \vee c \leqq b \vee d \in I$. It follows $(a+b) \vee (c+d) - a \vee c \in I$, and therefore $(a+b) \vee (c+d) \in F + I$.

To show (b), consider $a \in (F+I)_+$. $a = a_1 + a_2$, $a_1 \in F$, $a_2 \in I$. Writing this $a_1^+ - a_1^- + a_2 \geqq 0$, we obtain $a_1^- \leqq a_1^+ + a_2 \leqq a_1^+ + a_2^+$. Since $a_1^- \wedge a_1^+ = 0$, it follows $a_1^- \leqq a_2^+$ and thus $a_1^- \in I$. Then $a = a_1^+ + (a_2 - a_1^-)$ is the desired decomposition.

(8.1) $C \oplus Ra$ *is a norm-closed sub-vector-lattice of* M.

Proof. That it is a sub-vector-lattice follows from (8.0). Suppose $\lim_n \|(f_n + g_n) - h\| = 0$, where $f_n \in C$, $g_n \in Ra$ for all n. Then $\{f_n + g_n\}$ is a Cauchy sequence. Applying (3.5), $\{f_n\}$ is also a Cauchy sequence, and therefore norm-converges to some $f \in C$. It follows $\{g_n\}$ norm-converges to $h - f$, and since Ra is norm-closed, $h - f \in Ra$. Thus, $h = f + (h - f) \in C \oplus Ra$, and we are through.

REMARK. It follows from (8.1) and the open-mapping theorem that the Banach space $C \oplus Ra$ is the topological direct sum of the Banach spaces C and Ra.

The following theorem and corollary are what we would expect almost from the very definition of Ra.

(8.2) *Given a net* $\{f_\alpha\}$ *in* $C \oplus Ra$, *if* $f_\alpha \downarrow 0$ *in* $C \oplus Ra$, *then* $f_\alpha \downarrow 0$ *in* M.

Proof. Let $f = \bigwedge_\alpha f_\alpha$-in-M and denote its components in $\bar{R}a$ and Ra' by g and h respectively. Suppose $g \neq 0$. Then there exists $r \in Ra$ such that $0 < r \leqq g$. But this gives $0 < r \leqq f$, which contradicts the fact that $\bigwedge_\alpha f_\alpha$-in-$C \oplus Ra = 0$. That $h = 0$ follows from (7.3) and the fact that $(C \oplus Ra)_{Ra'} = C_{Ra'}$.

COROLLARY. $L = \tilde{\Omega}(C \oplus Ra)$.

By the equal sign we mean again that the two sides are isomorphic and that the multiplication between L and $C \oplus Ra$ induced by the latter's imbedding in M gives this isomorphism.

Proof. As we know [7, (8.1)], $L \subset \Omega(C \oplus Ra)$ (in fact, L is a closed ideal there). From (8.2) above, $L \subset \tilde{\Omega}(C \oplus Ra)$; we prove the opposite inclusion. Consider $\phi \in \tilde{\Omega}(C \oplus Ra)$. Since $C \oplus Ra$ is a topological direct sum, we have $\phi = \psi + \omega$, $\psi = \phi | Ra$, $\omega = \phi | C$. Since Ra is an ideal in $C \oplus Ra$, ϕ inherits the continuity of ϕ; thus $\psi \in \tilde{\Omega}(Ra) \subset L$. And $\omega \in \Omega(C) = L$ (actually $\omega \in \tilde{\Omega}(C)$). It follows $\phi \in L$.

We turn to $S + Ra$. Our first theorem parallels a standard theorem in topology [9, §8, V].

(8.3) *The following subsets of* M *are identical*:

(a) $S + Ra$;

(b) $\{f | \delta(f) \in Ra\}$;

(c) *the set of elements of the form* $g + h$, *where* g *is u.s.c.* (g *is l.s.c.*) *and* $h \in Ra$.

Proof. Consider $f \in S + Ra$. $f = g + h$, $g \in S$, $h \in Ra$, hence from (2.17) and (3.6), $\delta(f) \leqq \delta(g) + \delta(h) \in Ra$. Next, consider f such that $\delta(f) \in Ra$. Then $f = l(f) + (f - l(f))$, with $0 \leqq f - l(f) \leqq \delta(f) \in Ra$; and also $f = u(f) - (u(f) - f)$, with $0 \leqq u(f) - f < \delta(f) \in Ra$. The elements described in (c) are of course in $S + Ra$, and we are through.

(8.4) $S + Ra$ *is a norm-closed sub-vector-lattice of* M.

Proof. That $S + Ra$ is a sub-vector-lattice follows from the discussion preceding (8.1). Now suppose $\lim_n \| f_n - f \| = 0$, with $\delta(f_n) \in Ra$ for all n. Then from (2.21) $\lim_n \| \delta(f_n) - \delta(f) \| = 0$, and the norm-closedness of Ra gives us that $\delta(f) \in Ra$.

REMARK 1. S + Ra corresponds to the pointwise discontinuous functions [9, §13, VI].

REMARK 2. In general, S itself is not norm-closed—indeed, we would be very interested in a description of its norm-closure. (8.4) gives us a partial result:

(8.5) *The norm-closure of* S *is contained in* S + Ra. *Thus for every* f *in this norm-closure,* $\delta(f) \in Ra$.

(8.2) and its corollary hold also for S + Ra:

(8.6) *Given a net* $\{f_\alpha\}$ *in* S + Ra, *if* $f_\alpha \downarrow 0$ *in* S + Ra, *then* $f_\alpha \downarrow 0$ *in* M.

The first part of the proof is the same as that for (8.2). The second part is even simpler, since instead of (7.3) we need only the fact that $S_{Ra'} = Ra'$ (7.2a).

COROLLARY $L = \tilde{\Omega}(S + Ra)$.

That $L \subset \tilde{\Omega}(S + Ra)$ follows by the same argument as was used for the corollary to (8.2). Conversely, consider $\phi \in \tilde{\Omega}(S + Ra)$, $\phi \geqq 0$. From the discussion in [8, §2], we can write $\phi = \psi + \omega$, $\psi \wedge \omega = 0$, where $\psi \in \Omega(Ra)$ and $\omega(Ra) = 0$. Then $\psi \in \tilde{\Omega}(Ra) \subset L$. ω coincides on C with an element μ of L; from the continuity of ω, it follows easily that they coincide on S; and from the positiveness of ω and μ, it follows $\mu(Ra) = 0$; hence ω and μ coincide on all of S + Ra, and are therefore identical. Thus ψ and ω are in L, and therefore $\phi \in L$.

REMARK. (8.6) and its corollary also hold for Ba + Ra, Bo + Ra, and U + Ra.

We note finally:

(8.7) $(S + Ra) \cap (Ra')_1 = 0$.

To establish this, consider $f \in (S + Ra) \cap (Ra')_1$, $f \geqq 0$, and write it $f = l(f) + (f - l(f))$. Since $0 \leqq l(f) \leqq f$, we have $0 \leqq f - l(f) \leqq f$, and thus $l(f)$ and $f - l(f)$ are also in $(Ra')_1$. But then $l(f) = 0$ from (7.1), and $f - l(f) = 0$ from (8.3).

9. **The cut-completion of** C. In this section we parallel Dilworth's characterization [2] of the completion by cuts, or normal completion, of C. It will of course be the quotient space (S + Ra)/Ra. It corresponds in set theory to Stone's characterization [13] of the Boolean algebra of sets with nowhere dense frontier modulo the nowhere dense sets.

For each $f \in M$, we define

$$f^\star = u(l(u(f))), \qquad f_\star = l(u(l(f))). \tag{9.1}$$

We give two alternate characterizations of $f^\star$ and $f_\star$. Let us call a subset A of C with the property:

$$g \geqq h \textit{ forall } h \leqq A \textit{ implies } g \in A \quad (g, h \in C)$$

an *upper Dedekind segment of* C (commonly called a *normal subset*); and analogously for a *lower Dedekind segment of* C. It is easily seen that every subset of C is contained in a unique smallest upper Dedekind segment of C and a unique smallest lower Dedekind segment of C.

Given $f \in M$, let $B = \{g \in C \mid g \geqq f\}$; then the unique smallest upper Dedekind segment of C containing B will be called the upper Dedekind segment of C *determined by* f. The lower Dedekind segment of C *determined by* f is defined similarly.

(9.2) *Given* $f \in M$,
(a) $f^\star = \bigwedge A$, *where* A *is the upper Dedekind segment determined by* f;
(b) $f^\star$ *is the smallest u.s.c. element* g *satisfying* $(f-g)^+ \in Ra$.

The verification of (a) is straightforward. We show (b). $u((f-f^\star)^+) = (u(f-f^\star))^+ \leqq (u(f)-l(f^\star))^+$, from (2.10) and (2.6). Now, it is easily verified that $l(f^\star) = l(u(f))$, giving us $u((f-f^\star)^+) \leqq \delta(u(f))$; applying (3.6), we obtain $(f-f^\star)^+ \in Ra$. Now, let g be any u.s.c. element such that $(f-g)^+ \in Ra$. Then $u((f-g)^+) \in Ra$, hence the inequalities $(l(u(f))-g)^+ \leqq (u(f)-g)^+ \leqq (u(f-g))^+ = u((f-g)^+)$ (from (2.6) and (2.10)) give us that $(l(u(f))-g)^+ \in Ra$. Since the last is l.s.c., it follows it must be 0, and thus $l(u(f)) \leqq g$. This gives $u(l(u(f))) \leqq g$, and we are through.

Similarly,

(9.3) *Given* $f \in M$,
(a) $f_\star = \bigvee A$, *where* A *is the lower Dedekind segment determined by* f;
(b) $f_\star$ *is the largest l.s.c. element* g *satisfying* $(g-f)^+ \in Ra$.

The next eight propositions cover the arithmetic of $f^\star$ and $f_\star$.

(9.4) (a) $f \leqq g$ *implies* $f^\star \leqq g^\star$, $f_\star \leqq g_\star$.
(b) *For* $\lambda \geqq 0$, $(\lambda f)^\star = \lambda f^\star$, $(\lambda f)_\star = \lambda f_\star$.
(c) $(-f)^\star = -f_\star$.
(d) $f^\star = (u(f))^\star$, $f_\star = (l(f))_\star$.
(e) $l(f) \leqq f_\star \leqq f^\star \leqq u(f)$.
(f) *If* f *is l.s.c.*, $f^\star = u(f)$; *if* f *is u.s.c.*, $f_\star = l(f)$.
(g) $f \in Ra$ *if and only if* $f^\star = f_\star = 0$.
(h) $f^{\star\star} = f^\star$, $f_{\star\star} = f_\star$.

(9.5) *For any bounded set* A *in* M,
(a) $(\bigwedge_{f \in A} f)_\star \leqq \bigwedge_{f \in A} f_\star \leqq \bigvee_{f \in A} f_\star \leqq (\bigvee_{f \in A} f)_\star$
(b) $(\bigwedge_{f \in A} f)^\star \leqq \bigwedge_{f \in A} f^\star \leqq \bigvee_{f \in A} f^\star \leqq (\bigvee_{f \in A} f)^\star$.

For a finite number of elements, the first inequality in (a) and the last in (b) become equalities:

$$(9.6)\qquad \begin{aligned}(f\wedge g)_\star &= f_\star \wedge g_\star,\\ (f\vee g)^\star &= f^\star \vee g^\star.\end{aligned}$$

$$(9.7)\qquad f_\star + g_\star \leqq (f+g)_\star \leqq f_\star + g^\star \leqq (f+g)^\star \leqq f^\star + g^\star.$$

(9.8) COROLLARY.

$$f_\star - g^\star \leqq (f-g)_\star < \begin{matrix} f^\star - g^\star \\ f_\star - g_\star \end{matrix} \leqq (f-g)^\star \leqq f^\star - g_\star.$$

Via (2.9) we obtain

(9.9) *Given* $f \in M$ *and* $g \in C$, *we have* $(f\vee g)^\star = f^\star \vee g$, $(f\wedge g)^\star) = f^\star \wedge g$, $(f+g)^\star = f^\star + g$, *and similarly with* * *replaced by* $_\star$.

COROLLARY. $(f^\star)^+ (=f^+)^\star$, $(f^\star)^- = (f^-)_\star$, $(f_\star)^+ = (f^+)_\star$, $(f_\star)^- = (f^-)^\star$.

If we denote by $\text{mid}(f, f^\star, f_\star)$ the element $(f \wedge f^\star) \vee f_\star = (f \vee f_\star) \wedge f^\star$ [16], then we can write $f = \text{mid}(f, f^\star, f_\star) + (f - f^\star)^+ - (f - f_\star)^-$; thus f differs from $\text{mid}(f, f^\star, f_\star)$ by a rare element. Also, it is not hard to show that $u(\text{mid}(f, f^\star, f_\star)) = f^\star$ and $l(\text{mid}(f,f^\star,f_\star)) = f_\star$; thus $f^\star - f_\star = \delta(\text{mid}(f, f^\star, f_\star))$.

(9.10) *Given* $f \in M$, *the following are equivalent*:
1° $f \in S + Ra$;
2° $\text{mid}(f, f^\star, f_\star) \in S + Ra$;
3° $f - f^\star \in Ra$;
4° $f - f_\star \in Ra$;
5° $f^\star - f_\star \in Ra$;
6° $f^\star = u(f_\star)$;
7° $f_\star = l(f^\star)$.

Following the terminology in topology, we will call $f \in M$ a *regular u.s.c. element* if $f = f^\star$, and similarly for regular *l.s.c. element*. It follows easily that

(9.11) *Given* $f \in M$, *the following are equivalent*:
1° f *is a regular u.s.c. element*;
2° $f = u(l(f))$;
3° $f = u(g)$ *for some l.s.c. element* g;
4° $f = \bigwedge A$ *for some upper Dedekind segment* A *of* C.
And analogously for a regular l.s.c. element.

In terms of the above, part of (9.10) can be stated in a simple form.

(9.12) *Given* $f \in M$, f *lies in* S + Ra *if and only if it differs from a regular u.s.c. element by an element of* Ra. *And u.s.c. can be replaced by l.s.c.*

We proceed to examine (S + Ra)/Ra. If I is an ideal in a vector lattice E and

π the canonical mapping of E onto the quotient vector space E/I, then E/I is itself a vector lattice under the order defined by taking πE_+ for positive cone, and π is a homomorphism. In particular, this holds for $\pi: S + Ra \to (S + Ra)/Ra$. Moreover, since S + Ra is a Banach lattice and Ra is norm-closed, (S + Ra)/Ra is also a Banach space, and in fact a Banach lattice.

(9.13) *π maps the set of regular u.s.c. elements onto* (S + Ra)/Ra, *and the mapping is one-one. Similarly, for the set of regular l.s.c. elements.*

Proof. We note first that

(i) $$\pi f^\star \leqq \pi g^\star \textit{ implies } f^\star \leqq g^\star.$$

For, the hypothesis means that $f^\star \leqq g^\star + h$ for some $h \in Ra$, hence $f^\star = f^{\star\star} \leqq (g^\star + h)^\star \leqq g^{\star\star} + h^\star = g^\star$ ((9.4), (9.7)). It follows from (i) that $\pi f^\star = \pi g^\star$ implies $f^\star = g^\star$, and thus the mapping is one-one. That it is onto follows immediately from (9.12).

Remark. It is also immediate from (9.10) that the unique $f^\star$ and the unique $g_\star$ in a given equivalence class are related to each other by $f^\star = u(g_\star) = g^\star$, $g_\star = l(f^\star) = f_\star$.

(9.14) *π is an isometry on the set of regular u.s.c. elements, and on the set of regular l.s.c. elements. It follows π is an isometric isomorphism on* C.

Proof. We have to prove that if f and g are regular l.s.c. elements (say), then $\|\pi(f-g)\| = \|f-g\|$, or equivalently, $\|(f-g)+h\| \geqq \|f-g\|$ for all $h \in Ra$. Assume for the present that $g \leqq f$. From the remark following (3.5), it is enough here to show that $l(f-g)$ differs from $f-g$ by an element of Ra and has the same norm. This follows from (cf. (2.14))

(i) $l(f-g) \leqq f-g \leqq u(l(f-g))$.

Only the second inequality requires proof. From (2.6), $u(l(f-g)) \geqq u(f-u(g)) \geqq f - l(u(g)) = f-g$.

Now suppose we do not have $g \leqq f$. $(f-g)^+ = f - f \wedge g$, and from (9.6), $f \wedge g$ is also a regular l.s.c. element; hence, from the first part of the proof, $\|\pi(f-g)^+\| = \|(f-g)^+\|$. Similarly, $\|\pi(f-g)^-\| = \|(f-g)^-\|$. Then, since (S + Ra)/Ra is a Banach lattice and π a homomorphism, $\|\pi(f-g)\| \geqq \|(\pi(f-g))^+\| = \|\pi(f-g)^+\| = \|(f-g)^+\|$ and, similarly, $\|\pi(f-g)\| \geqq \|(f-g)^-\|$. Thus, $\|\pi(f-g)\| \geqq \max(\|(f-g)^+\|, \|(f-g)^-\|) = \|f-g\|$.

We now verify that (S + Ra)/Ra is the cut-completion of C.

(9.15) (S + Ra)/Ra *is a complete vector lattice and* πC *is dense in it. Indeed, each element of* (S + Ra)/Ra *is the infimum of some subset of* πC (*and also the supremum of some subset of* πC).

LEMMA. *If* $f = \bigwedge A$, *where the elements of* A *are u.s.c.* (*hence* f *is also*), *then* $\pi f = \bigwedge \pi A$. *Similarly for a supremum of l.s.c. elements.*

Proof of the lemma. Suppose $\pi g \leqq \pi A$ for some $g \in S + Ra$; we show $\pi g \leqq \pi f$. It is enough (9.10) to show $g_\star \leqq f$. Well, from (9.12) and (f) of (9.4), $g_\star \leqq h_\star = l(h) \leqq h$ for all $h \in A$, hence, $g_\star \leqq \bigwedge A = f$.

(9.15) now follows from the lemma and (9.13).

We note finally, that π preserves the suprema and infima already existing in C:

(9.16) $\pi | C: C \to (S + Ra)/Ra$ *is continuous.*

Suppose $0 = \bigwedge A$-in-C, and let $f = \bigwedge A$. Then $f \in Ra$, hence $\pi f = 0$. But from the lemma above, $\pi f = \bigwedge \pi A$, and we are through.

(9.17) $\tilde{\Omega}(C) = \tilde{\Omega}((S + Ra)/Ra)$.

The equality is meant in the following sense. We have a multiplication $\mu(f)$ between $\tilde{\Omega}(C)$ and $S + Ra$, induced by their imbeddings in L and M. For a given $\mu \in \tilde{\Omega}(C)$, $\mu(f) = 0$ for all $f \in Ra$, hence μ has a constant value on each equivalence class in $S + Ra$ modulo Ra. Thus, we have a uniquely defined multiplication between $\tilde{\Omega}(C)$ and $(S + Ra)/Ra$; this multiplication gives an isomorphism of $\tilde{\Omega}(C)$ with $\tilde{\Omega}((S + Ra)/Ra)$.

We proceed to prove this. For simplicity, let us denote $(S + Ra)/Ra$ by $\mathscr{C}$. $S + Ra$ is separating on $\tilde{\Omega}(C)$, hence, $\mathscr{C}$ is also. It follows the above multiplication imbeds $\tilde{\Omega}(C)$ in $\Omega(\mathscr{C})$; and from (9.13), the imbedding is in $\tilde{\Omega}(\mathscr{C})$. We denote this imbedding by $j: \tilde{\Omega}(C) \to \tilde{\Omega}(\mathscr{C})$. It is completely described by $(j\mu)(\pi f) = \mu(f)$ for all $\mu \in \tilde{\Omega}(C)$ and $f \in C$.

For simplicity also, let us write π for $\pi | C$. We thus have $\pi: C \to \mathscr{C}$ and from it, $\pi^t: \Omega(\mathscr{C}) \to L$. π^t carries $\tilde{\Omega}(\mathscr{C})$ into $\tilde{\Omega}(C)$. For, consider $\nu \in \tilde{\Omega}(\mathscr{C}), \nu \geqq 0$, and suppose $f_\alpha \downarrow 0$ in C; then $\pi f_\alpha \downarrow 0$ in $\mathscr{C}$ (9.16), hence, $\inf_\alpha (\pi^t \nu)(f_\alpha) = \inf_\alpha \nu(\pi f_\alpha) = 0$, and so $\pi^t \nu \in \tilde{\Omega}(C)$.

We thus have

$$\tilde{\Omega}(C) \xrightarrow{j} \tilde{\Omega}(\mathscr{C}) \xrightarrow{\pi^t} \tilde{\Omega}(C).$$

Since $((\pi^t \circ j)\mu)(f) = (j\mu)(\pi f) = \mu(f)$ for all $\mu \in \tilde{\Omega}(C)$ and $f \in C$, $\pi^t \circ j$ is the identity map. We show j is onto and π^t is one-one, whence it will follow that j is an isomorphism. Given $\nu \in \tilde{\Omega}(\mathscr{C})$, ν defines μ on C by $\mu(f) = \nu(\pi f)$, and (9.16) gives us that $\mu \in \tilde{\Omega}(C)$; clearly $j\mu = \nu$, and thus j is onto. Given $\nu \in \tilde{\Omega}(\mathscr{C})$ such that $\pi^t \nu = 0$, we have $\nu(\pi C) = (\pi^t \nu)(C) = 0$; since πC is dense in $\mathscr{C}$, it follows $\nu = 0$, and thus π^t is one-one.

For $X = [0,1]$, $\tilde{\Omega}(C) = 0$ (§4), hence

COROLLARY. *The cut-completion of* C[0,1] *has no continuous linear functionals.*

We note finally (cf. (7.2a) and the remark following it) that

(9.18) *If* $\tilde{\Omega}(C)$ *is separating on* C, *then* Ra′ *is isomorphic with the cut-completion of* C.

10. **The ideal Me of meager elements.** The σ-closure of Ra will be denoted by Me, and its elements will be called *meager*. We have immediately,

(10.1) Me *is a σ-closed, hence norm-closed, ideal, and* $\mathrm{Ra} \subset \mathrm{Me} \subset \overline{\mathrm{Ra}}$.

Me is in general strictly larger than Ra, that is, $f \in \mathrm{Me}$ does not imply $l(u(f)) = 0$. However, we still can draw two strong conclusions from the statement that $f \in \mathrm{Me}$. The first is simply the Category Theorem (3.3) restated:

(10.2) *For all* $f \in \mathrm{Me}$, $l(f) \leqq 0 \leqq u(f)$.

COROLLARY 1. $\mathrm{Me} \cap C = 0$.

COROLLARY 2. $\mathrm{Me} \cap S \subset \mathrm{Ra}$.

To show this last, consider $f \in \mathrm{Me} \cap S$, $f \geqq 0$, and write $f = l(f) + (f - l(f))$. Then $l(f) = 0$ from (10.2) and $f - l(f) \in \mathrm{Ra}$ from (3.6).

We turn to the second conclusion. In general, $f \in \overline{\mathrm{Ra}}$ does not imply $u(f) \in \overline{\mathrm{Ra}}$ (cf. (b) of (4.9)). Such an implication does hold for $f \in \mathrm{Ra}$ by the very definition of Ra. It also holds for $f \in \mathrm{Me}$:

(10.3) *For all* $f \in \mathrm{Me}$, $u(f) \in \overline{\mathrm{Ra}}$.

Proof. Assume first that $f \geqq 0$. $f = \bigvee_{n=1}^{\infty} g_n$, $g_n \in \mathrm{Ra}$, $g_n \geqq 0$ $(n = 1, 2, \cdots)$. Since $u(g_n) \in \mathrm{Ra}$ also, we can replace f by $\bigvee_{n=1}^{\infty} u(g_n)$; hence for simplicity, we will assume the g_n's are all u.s.c. To show $u(f) \in \overline{\mathrm{Ra}}$, we show that for every $\mu \in \tilde{\Omega}(C)$, $\mu \geqq 0$, we have $\mu(u(f)) = 0$. Let μ be any such μ, and consider $\varepsilon > 0$. From [6, (6.10)] and the fact that $\mu(f) = 0$, there exists g l.s.c. such that $g \geqq f$ and $\mu(g) \leqq \varepsilon$. Then $\mu(u(f)) \leqq \mu(u(g)) = \mu(g)$ (3.6) $\leqq \varepsilon$. Since ε was arbitrary, it follows $\mu(u(f)) = 0$. For general $f \in \mathrm{Me}$, we have (2.11) $u(f) = u(f^+) - l(f^-) \in \overline{\mathrm{Ra}}$, from the first part of the proof.

COROLLARY. *If* f *is in* Me, *then* $u(f)$, $l(f)$, *and* $\delta(f)$ *are all in* $\overline{\mathrm{Ra}}$.

Ra and Me correspond, of course, to the families of nowhere dense sets and sets of first category in topology. However, it is possible to have $f \in \mathrm{Ra}$ while the set $\{x \in X \mid f(x) \neq 0\}$ is of first category but not nowhere dense. As an example, let $X = [0,1]$, $A = \{x_n\}$ the set of rational numbers in X, and f the element of M_0 defined by $f(x_n) = 1/n$ $(n = 1, 2, \cdots)$, $f(x) = 0$ otherwise. Then $f \in \mathrm{Ra}$, but A is of first category without being nowhere dense. This corresponds to the fact that $\mathbf{1}_f$ is in Me but not in Ra (since $u(\mathbf{1}_f)=\mathbf{1}$). By way of illustration, this divergence from set theory shows up in making more precise a simple theorem in function theory. Given f l.s.c., the function-theoretic theorem states that $u(f)$ differs from f on X on at most a set of first category [3, Lemma 4]; (3.6) tells us more: that $u(f)-f \in \mathrm{Ra}$.

We turn to a comparison of Me and Ra with respect to the specific properties of Ra already developed in the paper. (3.4) and (3.5), and the remark following (3.5), carry over unchanged to Me (cf. (10.2)). Property (a) of (3.6) does not carry over: in the paragraph above, $\mathbf{1}_f \in \mathrm{Me}$ while $\delta(\mathbf{1}_f)=\mathbf{1}$.

From the corollary of (5.4), we have

(10.4) *If* $\tilde{\Omega}(C)$ *is separating on* C, *then* Me = Ra.

The inclusions of (10.1) give us that $\tilde{\Omega}(\mathrm{Me})=\tilde{\Omega}(\mathrm{Ra})$. Corollary 1 of (10.1) gives us that C + Me is a direct sum, $C \oplus \mathrm{Me}$, and the same argument as was used for (8.1) gives us that it is a norm-closed sub-vector-lattice of M. Finally, the argument for (8.2) and its corollary give us again that (i) if $f_\alpha \downarrow 0$ in $C \oplus \mathrm{Me}$, then $f_\alpha \downarrow 0$ in M, and (ii) $L=\tilde{\Omega}(C \oplus \mathrm{Me})$.

Now, consider S + Me. It corresponds in topology to the family of sets with the Baire property [9, §§11, 28]. And, as in topology, we have

(10.5) *Given* $f \in S+\mathrm{Me}$, *we can write* $f=g+h$, *where* g *is l.s.c. and* $h \in \mathrm{Me}$.

For, $f=f_1+f_2$, $f_1 \in S$, $f_2 \in \mathrm{Me}$, hence $f=l(f_1)+(f_2+f_1-l(f_1))$ is the desired decomposition (3.6).

(10.5) is of course the analogue of (c) in (8.3). In contradistinction to S + Ra, we have

(10.6) S + Me *is* σ*-closed* (*hence norm-closed*).

Proof. It is enough to show that given $\{h_n\} \subset (S+\mathrm{Me})_+$, if $h=\bigvee_n h_n$, then $h \in S+\mathrm{Me}$. For each n, $h_n=f_n+g_n$, $f_n \in S$, $g_n \in \mathrm{Me}$; from (8.0), we can take $f_n \geqq 0$, and from (10.5), we can replace it by $l(f_n)$, hence for simplicity we can assume f_n is a positive l.s.c. element. Since $\{h_n\}$ is bounded, it follows from (a) of (3.4) that $\{f_n\}$ is also bounded, and a fortiori, $\{g_n\}$ is too. Let $f=\bigvee_n f_n$; f is of course l.s.c.

(i) $$\bigwedge_n g_n \leqq h-f \leqq \bigvee_n g_n.$$

For, $\bigwedge_n g_n=\bigwedge_n(h_n-f_n) \leqq \bigwedge_n(h-f_n)=h-\bigvee_n f_n=h-f$, and $\bigvee_n g_n=\bigvee_n(h_n-f_n) \geqq \bigvee_n(h_n-f)=\bigvee_n h_n-f=h-f$. Combining (i) with (10.1), we have $h-f \in \mathrm{Me}$, whence $h=f+(h-f) \in S+\mathrm{Me}$.

It follows from (10.6), just as in topology, that

(10.7) Corollary. $\mathrm{Bo} \subset S+\mathrm{Me}$, *hence* $S+\mathrm{Me}=\mathrm{Bo}+\mathrm{Me}$.

And also, just as in topology [17, p. 178],

(10.8) $(\mathrm{Bo}+\mathrm{Me})/\mathrm{Me} = (\mathrm{S}+\mathrm{Ra})/\mathrm{Ra}$.

For, $(\mathrm{Bo}+\mathrm{Me})/\mathrm{Me} = (\mathrm{S}+\mathrm{Me})/\mathrm{Me} = \mathrm{S}/\mathrm{S}\cap\mathrm{Me} = \mathrm{S}/\mathrm{S}\cap\mathrm{Ra} = (\mathrm{S}+\mathrm{Ra})/\mathrm{Ra}$, the third equality following from Corollary 2 to (10.2).

The σ-closedness of Me in M makes it also σ-closed in any σ-closed subset of M containing Me; in particular

(10.9) Me *is a σ-closed ideal in* $\mathrm{S}+\mathrm{Me}$.

The arguments used in §8 (together with (10.7)) also give us

(10.10) *Given a net* $\{f_\alpha\}$ *in* $\mathrm{S}+\mathrm{Me}$, *if* $f_\alpha\downarrow 0$ *in* $\mathrm{S}+\mathrm{Me}$, *then* $f_\alpha\downarrow 0$ *in* M.

COROLLARY. $\mathrm{L} = \tilde{\Omega}(\mathrm{S}+\mathrm{Me})$.

As in (8.17),

(10.11) $(\mathrm{S}+\mathrm{Me})\cap(\mathrm{Ra}')_1 = 0$.

For this it is sufficient to note that in the proof of (8.7) we actually only need that $\delta(f)\in\overline{\mathrm{Ra}}$ (not Ra). This is supplied for us by the corollary to (10.3).

We again remark that (10.10) and its corollary hold for $\mathrm{U}+\mathrm{Me}$.

The regular u.s.c. element f^* determined by an element f of M has been characterized by three equivalent properties: the one used to define it in (9.1), and the two properties in (9.2). The question arises, does each $f\in\mathrm{M}$ determine an analogous element in terms of Me? We show

(10.12) *Given* $f\in\mathrm{M}$, *there exists a smallest u.s.c. element g satisfying* $(f-g)^+\in\mathrm{Me}$.

This is of course similar to (b) of (9.2). We do not know any characterization of this smallest u.s.c. element similar to (9.1) or (a) of (9.2). Our proof takes up the remainder of this section. It is based on Kuratowski's proof of the corresponding theorem in topology [9, §10, Théorème III]. We do this by reducing the problem to one involving components of **1**, where Kuratowski's argument can be paralleled. It seems to us a direct proof (not involving components of **1**) should be possible, but we have been unable to obtain one.

LEMMA. *Given a family* $\{g_\alpha\}$ *of l.s.c. elements and* $f\in\mathrm{M}$, *if* $f\wedge g_\alpha\in\mathrm{Me}$ *for all* α, *then* $\bigvee_\alpha(f\wedge g_\alpha)\in\mathrm{Me}$.

A proof of this lemma is given in the Appendix (II). We proceed to prove the theorem.

(I) *Given a component* e *of* **1**, *there exists a smallest u.s.c. component* d *of* **1** *satisfying* $(e-d)^+\in\mathrm{Me}$ (*that is, the theorem holds for components of* **1**).

Let $\{d_\alpha\}$ be the set of u.s.c. components of $\mathbf{1}$ with the above property. Denoting $\mathbf{1}-d_\alpha$ by e_α, we have that e_α is l.s.c. and $e\wedge e_\alpha=(e-d_\alpha)^+$. It follows from the Lemma that $\bigvee_\alpha e\wedge e_\alpha\in Me$. Thus, $(e-\bigwedge_\alpha d_\alpha)^+=(\bigvee_\alpha(e-d_\alpha))^+=\bigvee_\alpha(e-d_\alpha)^+$ $=\bigvee_\alpha e\wedge e_\alpha\in Me$, and $\bigwedge_\alpha d_\alpha$ is the desired d.

Turning to a general $f\in M$, we can assume $0\leqq f\leqq \mathbf{1}$. For each $0\leqq\lambda\leqq 1$ let $e(\lambda)=\mathbf{1}_{(f-\lambda 1)^+}$ and let $d(\lambda)$ be the corresponding u.s.c. component of $\mathbf{1}$ given by (I). It is easily verified that

(i) $\lambda_1\leqq\lambda_2$ *implies* $d(\lambda_1)\geqq d(\lambda_2)$.

Now for each $n=1,2,\cdots$, define

$$g_n=\bigvee_{k=1}^{2^n}\frac{k}{2^n}d\left(\frac{k}{2^n}\right),\qquad h_n=\bigvee_{k=1}^{2^n}\frac{k}{2^n}d\left(\frac{k-1}{2^n}\right).$$

(II) *The two sequences* $\{g_n\}$, $\{h_n\}$ *have the following properties*:

(ii) g_n *and* h_n *are u.s.c. for every* n;

(iii) $g_1\leqq g_2\leqq g_3\leqq\cdots\leqq h_3\leqq h_2\leqq h_1$;

(iv) $\|h_n-g_n\|\leqq 1/2^n$.

(ii) is clear; we show (iii).

$$h_{n+1}=\bigvee_{k=1}^{2^{n+1}}\frac{k}{2^{n+1}}d\left(\frac{k-1}{2^{n+1}}\right)=\bigvee_{j=1}^{2^n}\left\{\left[\frac{2j-1}{2^{n+1}}d\left(\frac{2j-2}{2^{n+1}}\right)\right]\vee\left[\frac{2j}{2^{n+1}}d\left(\frac{2j-1}{2^{n+1}}\right)\right]\right\}.$$

Straightforward computation shows that the expression in each bracket is $\leqq(j/2^n)d((j-1)/2^n)$, and thus $h_{n+1}\leqq\bigvee_{j=1}^{2^n}(j/2^n)d((j-1)/2^n)=h_n$. Again

$$g_{n+1}=\bigvee_{k=1}^{2^{n+1}}\frac{k}{2^{n+1}}d\left(\frac{k}{2^{n+1}}\right)=\left[\bigvee_{j=1}^{2^n}\frac{2j}{2^{n+1}}d\left(\frac{2j}{2^{n+1}}\right)\right]\vee\left[\bigvee_{j=1}^{2^n}\frac{2j-1}{2^{n+1}}d\left(\frac{2j-1}{2^{n+1}}\right)\right].$$

Since the expression in the first bracket is g_n, we have $g_{n+1}\leqq g_n$. That $g_n\leqq h_n$ for every n follows directly from (i).

We now show (iv).

$$\begin{aligned}h_n-g_n&=\bigvee_{k=1}^{2^n}\frac{k}{2^n}d\left(\frac{k-1}{2^n}\right)-\bigvee_{k=1}^{2^n}\frac{k}{2^n}d\left(\frac{k}{2^n}\right)\\&=\bigvee_{k=1}^{2^n}\left[\frac{k}{2^n}d\left(\frac{k-1}{2^n}\right)-\bigvee_{j=1}^{2^n}\frac{j}{2^n}d\left(\frac{j}{2^{n+1}}\right)\right]\\&\leqq\bigvee_{k=1}^{2^n}\left[\frac{k}{2^n}d\left(\frac{k-1}{2^n}\right)-\frac{k-1}{2^n}d\left(\frac{k-1}{2^n}\right)\right]\\&=\bigvee_{k=1}^{2^n}\frac{1}{2^n}d\left(\frac{k-1}{2^n}\right)\leqq\frac{1}{2^n}\mathbf{1}.\end{aligned}$$

This completes the proof of (II).

It follows from (II) that $\bigvee_n g_n=\bigwedge_n h_n$, and this common element is u.s.c.; we denote it by h.

(III) $(f-h)^+\in Me$.

$(f-h)^+ = \bigvee_n (f-h_n)^+$, and thus we need only show $(f-h_n)^+ \in Me$ for each n. Noting that $e(0)=\mathbf{1}_f$, hence $f_{e(0)}=f$, we have $(f-h_n)^+ \leqq (f-(1/2^n)d(0))^+$ (definition of h_n) $=(f_{d(0)}+f_{(e(0)-d(0))^+}-(1/2^n)d(0))^+ \leqq (f_{d(0)}-(1/2^n)d(0))^+ + (e(0)-d(0))^+$. Since $d(0)=\mathbf{1}_{d(0)}$, the first term in this last expression becomes $[(f-(1/2^n)\mathbf{1})^+]_{d(0)} \leqq (f-(1/2^n)\mathbf{1})^+ \leqq e(1/2^n) \leqq d(1/2^n)+(e(1/2^n)-d(1/2^n))^+$. We thus obtain $(f-h_n)^+ \leqq d(1/2^n)+[(e(1/2^n)-d(1/2^n))^+ + (e(0)-d(0))^+]$, where the term in brackets is an element of Me (I) and is disjoint from $d(1/2^n)$.

Assume we have showed that for $k<2^n-1$, $(f-h_n)^+ \leqq d(k/2^n)+m(k)$, where m(k) is an element of Me disjoint from $d(k/2^n)$. We show that then a similar inequality holds with k replaced by k + 1. To do this, it is enough to work with $[(f-h_n)^+]_{d(k/2^n)}$, since $[(f-h_n)^+]_{m(k)} \in Me$. Well, $[(f-h_n)^+]_{d(k/2^n)} \leqq [(f-((k+1)/2^n)d(k/2^n))^+]_{d(k/2^n)}$ (definition of h_n) $=[(f-((k+1)/2^n)\mathbf{1})^+]_{d(k/2^n)} \leqq e((k+1)/2^n) \leqq d((k+1)/2^n)+(e((k+1)/2^n)-d((k+1)/2^n))^+$, with the last term in Me.

Repeating the process 2^n-2 times gives us finally that

$$(f-h_n)^+ \leqq d((2^n-1)/2^n)+m(2^n-1),$$

where $m(2^n-1) \in Me$. We complete the proof of (III) by showing that $(f-h_n)^+ \wedge d((2^{n-1})/2^n)=0$, whence $(f-h_n)^+ \leqq m(2^n-1) \in Me$. $(f-h_n)^+ \wedge d((2^{n-1})/2^n)=[(f-h_n)^+]_{d((2^n-1)/2^n)} = (f_{d((2^n-1)/2^n)}-(h_n)_{d((2^n-1)/2^n)})^+ = (f_{d((2^n-1)/2^n)}-d((2^n-1)/2^n))^+$ (definition of h_n) $=[(f-\mathbf{1})^+]_{d((2^n-1)/2^n)}=0$, since $f \leqq \mathbf{1}$.

To establish (10.12), it remains only to show

(IV) *If* g *is a u.s.c. element satisfying* $(f-g)^+ \in Me$, *then* $g \geqq h$.

Since $h=\bigvee_n g_n$, it is enough to show that for any n and $1 \leqq k \leqq 2^n$, $g \geqq (k/2^n)d(k/2^n)$ (definition of g_n). For simplicity of notation, we denote $k/2^n$ by λ; thus $d(k/2^n)=d(\lambda)$. Assume the desired inequality does not hold. $(\lambda\mathbf{1}-g)^+$ is l.s.c., hence $c=\mathbf{1}_{(\lambda 1-g)^+}$ is l.s.c. also. $d(\lambda)-d(\lambda)\wedge c$ is then u.s.c.; denote it by b.

(v) $b<d(\lambda)$.

Suppose $b=d(\lambda)$. Then $d(\lambda)\wedge c=0$. Now $(\lambda d(\lambda)-g)^+ \leqq (\lambda\mathbf{1}-g)^+ \leqq c$; since $(\lambda d(\lambda)-g)^+ \leqq \lambda d(\lambda) \leqq d(\lambda)$, it follows $(\lambda d(\lambda)-g)^+ \leqq d(\lambda)\wedge c=0$. This says $g \geqq \lambda d(\lambda)$, contradicting our assumption on g; thus (v) holds. We show $(e(\lambda)-b)^+ \in Me$, which will contradict the definition of $d(\lambda)$. $e(\lambda)=e(\lambda)\wedge d(\lambda)+(e(\lambda)-d(\lambda))^+$, with the last term lying in Me, hence, we need only show $(e(\lambda)\wedge d(\lambda)-b)^+ \in Me$.

$$\begin{aligned}(e(\lambda)\wedge d(\lambda)-b)^+ &\leqq e(\lambda)\wedge(d(\lambda)-b)\\ &= (d(\lambda)-b)_{e(\lambda)}\\ &= (d(\lambda)-b)_{(f-\lambda 1)^+}\\ &= (d(\lambda)-b)_{(f_{(d(\lambda)-b)}-\lambda(d(\lambda)-b))^+}\\ &\leqq (d(\lambda)-b)_{(f_{(d(\lambda)-b)}-g_{(d(\lambda)-b)})^+},\end{aligned}$$

the last inequality coming from the inequality, $\lambda(d(\lambda)-b) \geqq g_{(d(\lambda)-b)}$, which can be verified by routine computation. Now

$$(d(\lambda)-b)_{(f_{(d(\lambda)-b)}-g_{(d(\lambda)-b)})^+} = (d(\lambda)-b)_{(f-g)^+{}_{(d(\lambda)-b)}} \leqq \mathbf{1}_{(f-g)^+} \in \mathrm{Me},$$

and we are through.

11. **When C is a complete vector lattice.** The material in this section stems principally from Dixmier's paper [3] (cf. our earlier discussion in §5).

In line with our general rule on terminology, we write simply "complet" for "vector-lattice complete."

(11.1) *The following statements are equivalent:*

1° C *is complete;*

2° f *l.s.c. implies* $u(f) \in C$;

3° *every regular u.s.c. element lies in* C;

4° $f^* \in C$ *for every* $f \in M$.

And of course 2°, 3°, 4° *can be replaced by the statements obtained in interchanging u.s.c. and l.s.c.,* $u(f)$ *and* $l(f)$, *and* f^* *and* f_*.

Proof. That 2°, 3°, and 4° are equivalent follows from (9.11). We show 1° implies 2°. Consider an l.s.c. element f and let $A = \{g \in C \mid g \leqq f\}$, whence $f = \bigvee A$. Since C is complete, there exists $h = \bigvee A$-in-C. Then, $h \geqq f$ and is the smallest element of C with this property. It follows $h = u(f)$. That 2° implies 1° is immediate.

Corollary. C *is complete if and only if every element of* S *differs from an element of* C *by an element of* Ra. *In which case,*

(a) $C + Ra = S + Ra$,

(b) $C + Me = Bo + Me$.

The second of these follows from (10.7).

(11.1) corresponds to the Stone-Nakano Theorem that C is complete if and only if X is extremally disconnected [14], [11] (cf. [3, §1]). For the corollary, cf. Theorem 9 in [14].

Another corollary of (11.1), via (9.13), is

(11.2) *If* C *is complete, it is isomorphic with* $(S + Ra)/Ra$.

(This of course also follows from the fact that $(S + Ra)/Ra$ is isomorphic with the cut-completion of C.)

As we know (10.8), $(S + Ra)/Ra$ is isomorphic with $(Bo + Me)/Me$. We digress here to note

(11.3) *If* C *is* σ*-complete, it is isomorphic with* $(Ba + Me)/Me$.

The interest in this theorem is that it corresponds to the Loomis-Sikorski Theorem that a σ-complete Boolean algebra is isomorphic to a σ-closed field of subsets of its Stone space modulo a σ-closed ideal of subsets [15, §29]. We need only show that if C is σ-complete, C + Me is σ-closed. This will give (11.3), since then C + Me = Ba + Me, and the isomorphism of C with (C + Me)/Me is immediate.

Consider $\{h_n\} \subset (C + Me)_+$, with $h = \bigvee_n h_n$. For each n, $h_n = f_n + g_n$, $f_n \in C$, $g_n \in Me$; from (8.0), we can take $f_n \geqq 0$. Since $\{h_n\}$ is bounded, it follows from (3.4) that $\{f_n\}$ is also bounded, hence a fortiori $\{g_n\}$ is too. Let $f = \bigvee_n f_n$; then $u(f) \in C$ (since C is σ-complete, the argument is the same as for (11.1)), and $u(f) - f \in Ra \subset Me$. As in the proof of (10.6), $\bigwedge_n g_n \leqq h - f \leqq \bigvee_n g_n$, hence,

$$\bigwedge_n g_n - (u(f) - f) \leqq h - u(f) \leqq \bigvee_n g_n - (u(f) - f).$$

Since the first and last terms are in Me, $h - u(f)$ is also; we thus have $h = u(f) + (h - u(f)) \in C + Me$, and we are through.

We return to the case where C is complete. It follows easily from (7.2a) and (11.1) that

(11.4) *If* C *is complete*, $C_{Ra'} = Ra'$.

One consequence of this is that if C is complete, $C_{Ra'}$ is reflexive with respect to continuity, since Ra′ is (we say E is *reflexive with respect to* (*order*) *continuity* if $\tilde{\Omega}(E)$ is separating on E and the canonical imbedding of E in $\tilde{\Omega}^2(E)$ is onto). If $\tilde{\Omega}(C)$ is separating on C, then $C_{Ra'}$ is isomorphic with C. Thus (cf. the corollary to Theorem 1 in [3]):

(11.5) *The following statements are equivalent*:

1° C *is complete and* $\tilde{\Omega}(C)$ *is separating on* C (*i.e.*, X *is hyperstonian* [3]);

2° C *is reflexive with respect to continuity*;

3° $C_{Ra'} = Ra'$.

We strengthen two of the theorems in §5. (5.3) becomes

(11.6) *If* C *is complete, then for each* $\mu \in \tilde{\Omega}(C)$, $M_\mu = C_\mu$. *More explicitly, given* $h \in M_\mu$, $h \geqq 0$, *then* $u(h) \in C$ *and* $(u(h))_\mu = h$.

The first statement here is a corollary of (11.4), but for the entire theorem we need more detail. From the corollary to (5.2), $(u(h))_\mu = h$. We show $u(h) \in C$. (5.2) gives us that $(l(u(h)))_\mu = h$, hence $h \leqq l(u(h)) \leqq u(h)$. Since $l(u(h)) \in C$, it follows $l(u(h)) = u(h)$, and we are through.

REMARK. In particular, $u(1_\mu) \in C$, and thus the support of μ is both open and closed [12, Satz 11.4], [3, Proposition 3].

The corollary to (11.1) combined with (10.4) gives us

(11.7) *If* C *is complete and* $\tilde{\Omega}(C)$ *is separating on* C, *then* C + Ra = Bo + Me.

This in turn gives us a strengthening of (5.5) [3, Proposition 6]:

(11.8) *If* C *is complete and* $\tilde{\Omega}(C)$ *is separating on* C, *then*

$$\bigcap_{\mu \in \tilde{\Omega}(C)} \mathcal{M}(\mu) = C + Ra = Bo + Me.$$

12. **More on homomorphisms of** C. We saw in (6.7) that, given a homomorphism $h:C(X)\to C(Y)$, h^{tt} carries each l.s.c. element of M(X) into an l.s.c. element of M(Y). If h is an isomorphism, we can show the following partial converse (we denote both $\mathbf{1}(X)$ and $\mathbf{1}(Y)$ simply by $\mathbf{1}$).

(12.1) *Let* $h:C(X)\to C(Y)$ *be an isomorphism which satisfies*: $h\mathbf{1}=\mathbf{1}$. *Then for* $f\in U(X)$, $h^{tt}f$ *l.s.c. implies* f *l.s.c.*

Proof. We can assume $0\leqq f\leqq \mathbf{1}$. We use two general lemmas which are not hard to show.

LEMMA 1. *Given* $f\in U$, *the following are equivalent*:

1° f *is l.s.c.*;

2° f *is lower semicontinuous on* L_+ *under* w(L,C);

3° f *is lower semicontinuous on* X.

LEMMA 2. *Given* f *l.s.c.*, $0\leqq f\leqq \mathbf{1}$.

(a) *For each* $\lambda\geqq 0$, $\mathbf{1}_{(f-\lambda\mathbf{1})^+}$ *is l.s.c.*

(b) $\| f-1/n\ \sum_{k=1}^n \mathbf{1}_{(f-(k/n)\mathbf{1})^+}\| \leqq 1/n \quad (n=1,2,\cdots)$.

We proceed to prove the theorem. We note first that, since in any C, $\|f\| = \inf\{\lambda \mid \lambda\mathbf{1}\geqq |f|\}$,

(i) h *is an isometry.*

We next reduce the theorem to the case where f is a component of $\mathbf{1}$.

(ii) *For any* $g\in M(X)$, $h^{tt}\mathbf{1}_g = \mathbf{1}_{h^{tt}g}$.

For, taking $g\geqq 0$ for simplicity, since h^{tt} is a continuous homomorphism, $h^{tt}\mathbf{1}_g = h^{tt}(\bigvee_n \mathbf{1}\wedge ng) = \bigvee_n h^{tt}(\mathbf{1}\wedge ng) = \bigvee_n (h^{tt}\mathbf{1}\wedge h^{tt}ng) = \bigvee_n(\mathbf{1}\wedge nh^{tt}g) = \mathbf{1}_{h^{tt}g}$. It follows from (ii) that for every $\lambda\geqq 0$, $h^{tt}\mathbf{1}_{(f-\lambda\mathbf{1})^+} = \mathbf{1}_{h^{tt}(f-\lambda\mathbf{1})^+} = \mathbf{1}_{(h^{tt}f-\lambda\mathbf{1})^+}$. Combining this with (i), Lemma 2, and the fact that the set of l.s.c. elements is norm-closed, we obtain the desired conclusion that we need only prove (12.1) for the case where f is a component of $\mathbf{1}$.

$h^{tt}f$ is then also a component of $\mathbf{1}$, for $h^{tt}f\wedge(\mathbf{1}-h^{tt}f) = h^{tt}f\wedge h^{tt}(\mathbf{1}-f) = h^{tt}(f\wedge(\mathbf{1}-f)) = 0$. Thus f takes on only the values 0, 1 on X, and $h^{tt}f$ takes on only the values 0, 1 on Y. Set $A=\{x\mid f(x)=0\}$ and $B=\{y\mid (h^{tt}f)(y)=0\}$; since h^t carries Y onto X (6.9), $A=h^tB$. Now $h^{tt}f$ is lower semicontinuous on Y (Lemma 1), hence B is compact, hence A is closed, hence f is lower semicontinuous on X, and thus (again from Lemma 1) f is l.s.c.

In our second paper, we stated the following [7, (8.8)]:

(12.2) *Every l.s.c. element in* Ba *is the supremum of a countable subset of* C (*and similarly for every u.s.c. element in* Ba).

Our "proof" there is completely erroneous. The following argument (indeed the theorem itself) is an adaptation from Halmos [5, §51, Theorem D].

Given a set A in M, let us denote by σA the σ-closure of A, that is, the smallest σ-closed set in M containing A. Now consider g l.s.c., $g\in Ba$. By a standard argument [5, §5, Theorem D], $g\in\sigma A$ where A is a countable subset of C, and we can assume $\mathbf{1} \in A$. Then the norm-closed sub-vector-lattice generated by A is an (M)-space which is separable in the norm topology, hence can be written as C(Z), where Z is a compact metric space.

Let $i: C(Z) \to C = C(X)$ be the identity mapping. Then i is an isomorphism with $i\mathbf{1} = \mathbf{1}$, and we can apply (12.1). Now $A \subset iC(Z) = i^{tt}C(Z)$, so from the continuity of i^{tt}, $\sigma A \subset i^{tt}\mathrm{Ba}(Z)$. Applying (12.1), $g = i^{tt}f$, where f is an l.s.c. element of M(Z). Since Z is compact metric, $f = \bigvee_n f_n, \{f_n\} \subset C(Z)$. It follows—again from the continuity of i^{tt}—that $g = \bigvee_n i^{tt}f_n$, and since $i^{tt}C(Z) = iC(Z) \subset C(X)$, we are through.

Appendix I

Proof of (4.2). Replacing E by $\bar{I}$ if necessary, we can assume I is dense in E; then what we need to show is that $\tilde{\Omega}(I) \cap \Omega(E) = \tilde{\Omega}(E)$. Since a continuous linear functional on E is a fortiori continuous on I, we have immediately that $\tilde{\Omega}(E) \subset \tilde{\Omega}(I) \cap \Omega(E)$. We show equality.

For the remainder of the proof, we denote the weak topology defined by I on $\Omega(I)$, $\Omega(E)$, or any of their subsets simply by w(I).

(i) $\tilde{\Omega}(E)$ *is* w(I)-*closed in* $\tilde{\Omega}(I) \cap \Omega(E)$.

In effect, from [7, (3.3)], the closure of $\tilde{\Omega}(E)$ in $\tilde{\Omega}(I)$ is w(I)-closed there. Since $\tilde{\Omega}(E)$ is already closed in $\tilde{\Omega}(I) \cap \Omega(E)$, we have (i). Now $\tilde{\Omega}(E)$ is separating on I, consequently is w(I)-dense in $\tilde{\Omega}(I)$, hence in $\tilde{\Omega}(I) \cap \Omega(E)$. Combining this with (i) gives us that $\tilde{\Omega}(E) = \tilde{\Omega}(I) \cap \Omega(E)$.

Appendix II

Proof of the Lemma in (10.12). We follow [9, § 10, Théorème III] for parts (a) and (b) of this proof.

(a) *Let* $\{\alpha\}$ *be the set of all ordinals* $<$ *some ordinal* η, *and* $\{g_\alpha\}$ *a set of l.s.c. elements bounded above. Set* $h_1 = g_1$, *and for* $\alpha > 1$, $h_\alpha = l((g_\alpha - \bigvee_{\beta<\alpha} g_\beta)^+)$. *Then* $\bigvee_\alpha g_\alpha \leqq u(\bigvee_\alpha h_\alpha)$.

It is enough (2.3) to show that for each $\alpha < \eta$, $g_\alpha \leqq u(\bigvee_{\beta \leqq \alpha} h_\beta)$. This is trivially true for $\alpha = 1$. Consider any $\alpha < \eta$, and suppose it is true for all $\beta < \alpha$. From (2.4), $u(\bigvee_{\beta\leqq\alpha} h_\beta) = u(\bigvee_{\beta<\alpha} h_\beta) \vee u(h_\alpha)$. Also

(i) $u(\bigvee_{\beta<\alpha} h_\beta) \geqq u(\bigvee_{\beta<\alpha} g_\beta) \geqq l(u(\bigvee_{\beta<\alpha} g_\beta))$;

(ii) $u(h_\alpha) \geqq [g_\alpha - l(u(\bigvee_{\beta<\alpha} g_\beta))]^+$.

(i) follows directly from the induction hypothesis. We show (ii). $h_\alpha = l((g_\alpha - \bigvee_{\beta<\alpha} g_\beta)^+) = (l(g_\alpha - \bigvee_{\beta<\alpha} g_\beta))^+$ (2.10) $\geqq (g_\alpha - u(\bigvee_{\beta<\alpha} g_\beta))^+$ (2.6). Hence $u(h_\alpha) \geqq (u(g_\alpha - u(\bigvee_{\beta<\alpha} g_\beta)))^+$ (2.10) $\geqq (g_\alpha - l(u(\bigvee_{\beta<\alpha} g_\beta)))^+$ (2.6). Since the supremum of the right side of (i) and the right side of (ii) is g_α, we have $u(\bigvee_{\beta\leqq\alpha} h_\beta) \geqq g_\alpha$, and therefore (a).

(b) *The lemma is true for the case where* f *and all the* g_α*'s are components of* $\mathbf{1}$.

Well-order the α's and define $\{h_\alpha\}$ as in (a). Since the g_α's are components of $\mathbf{1}$, the h_α's are mutually disjoint. Now from (a), $\bigvee_\alpha g_\alpha \leqq u(\bigvee_\alpha h_\alpha) \leqq \bigvee_\alpha h_\alpha + \delta(\bigvee_\alpha h_\alpha)$. This gives in turn, $\bigvee_\alpha (f \wedge g_\alpha) = f \wedge (\bigvee_\alpha g_\alpha) \leqq f \wedge (\bigvee_\alpha h_\alpha) + \delta(\bigvee_\alpha h_\alpha)$. Since $\bigvee_\alpha h_\alpha$ is l.s.c., $\delta(\bigvee_\alpha h_\alpha) \in \mathrm{Ra}$; we show $f \wedge (\bigvee_\alpha h_\alpha) \in \mathrm{Me}$. We can write it $\bigvee_\alpha f \wedge h_\alpha$. Each $f \wedge h_\alpha$ is in Me_+, hence can itself be written $f \wedge h_\alpha = \bigvee_n e_{\alpha n}$, $e_{\alpha n} \in \mathrm{Ra}_+ (n = 1, 2, \cdots)$. For each n, let $e_n = \bigvee_\alpha e_{\alpha n}$. Since the h_α's are disjoint, $e_n \wedge h_\alpha = e_{\alpha n}$ for all α; therefore, from (3.7), each e_n is in Ra. Then $\bigvee_\alpha (f \wedge h_\alpha) = \bigvee_n e_n \in \mathrm{Me}$, and (b) is proved.

Turning to the lemma, assume first that $f \geqq 0$ and $g_\alpha \geqq 0$ for all α; and for simplicity, let $f \leqq \mathbf{1}$. Then $\mathbf{1}_{g_\alpha}$ is l.s.c. for all α; and $\mathbf{1}_f \wedge \mathbf{1}_{g_\alpha} = \mathbf{1}_{f \wedge g_\alpha} \in \mathrm{Me}$ for all α. The inequality $\bigvee_\alpha f \wedge g_\alpha \leqq \bigvee_\alpha \mathbf{1}_{f \wedge g_\alpha} = \bigvee_\alpha \mathbf{1}_f \wedge \mathbf{1}_{g_\alpha}$ combined with (b), then gives us that $\bigvee_\alpha f \wedge g_\alpha \in \mathrm{Me}$.

For general f, $\{g_\alpha\}$ satisfying the hypothesis of the lemma, we note first that $[\bigvee_\alpha (f \wedge g_\alpha)]^+ = \bigvee_\alpha (f^+ \wedge g_\alpha^+)$, which is in Me from the immediately preceding argument. Then, choosing an arbitrary $\alpha_0, f \wedge g_{\alpha_0} \leqq \bigvee_\alpha (f \wedge g_\alpha) \leqq [\bigvee_\alpha (f \wedge g_\alpha)]^+$. Since the first and last of these are in Me, so is $\bigvee_\alpha (f \wedge g_\alpha)$, and we are through.

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